Problem for 2012 February

Proposed by Dan Jurca

Suppose $f : [0, \infty) \to \mathbf{R},\ 0 < f(0),\ f$ is differentiable in $(0, \infty)$, and $\lim_{x\to\infty} \dfrac{f(x)}{x} = \infty$.
Show that there exists a least m, say m_0, such that the line with equation $y = mx$ intersects the curve defined by $y = f(x)$, and the line $y = m_0 x$ is tangent to the curve at the point of intersection.

Solution by the proposer

Suppose $g : (0, \infty) \to \mathbf{R}$ by $g(x) = f(x)/x$; we show that the differentiable function g attains a minimum value. First it is clear that $\lim_{0^+} g = \lim_{\infty} g = \infty$. Therefore if $c = g(1) = f(1)$, then there exist $a \in \mathbf{R}$ and $b \in \mathbf{R}$ such that $0 < a < b,\ 0 < x \le a \ \Rightarrow\ c \le g(x)$, and $b \le x \ \Rightarrow\ c \le g(x)$. The function $g|_{[a,b]}$ (g restricted to the interval $[a, b]$) attains a minimum value, say m_0, in this interval, and obviously the minimum value of g, $g_{\min} = m_0$. Since g is a differentiable function this minimum occurs at a point x_0 where $g'(x_0) = 0$. It follows that

$$\begin{aligned} g'(x_0) &= \frac{f'(x_0) \cdot x_0 - f(x_0) \cdot 1}{x_0^2} \\ &= 0; \\ \text{therefore} \qquad f'(x_0) &= f(x_0)/x_0 \\ &= g(x_0) \\ &= g_{\min} \\ &= m_0. \end{aligned}$$

Now obviously the line with equation $y = m_0 x$ is tangent to the curve with equation $y = f(x)$ at the point $(x_0, f(x_0))$, since $f(x_0)/x_0 = f'(x_0)$. If $m < m_0$ and the line with equation $y = mx$ intersects the curve with equation $y = f(x)$ at the point $(t, f(t))$, then $mt = f(t)$, so that $g(t) = f(t)/t = m < m_0$, contradicting $g_{\min} = m_0$. Hence no line with slope less than m_0 intersects the curve defined by f.

Also solved by Kourosh Ghaderi and John M. Sayer