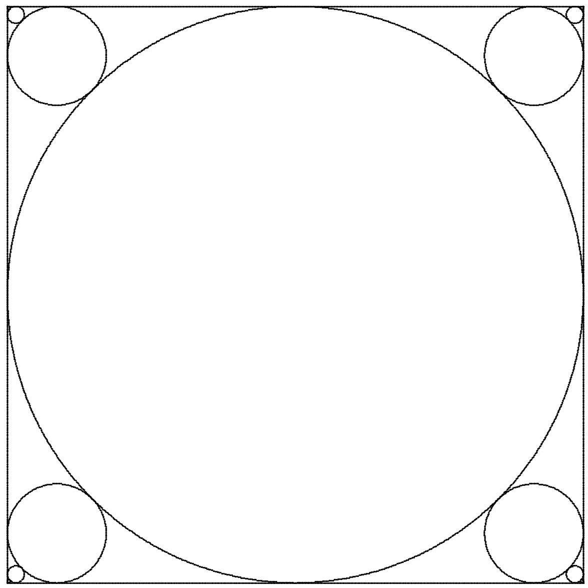
**Problem for 2012 April**

Proposed by Dan Jurca

A circle of radius 1 is inscribed in a square as shown in the sketch, and infinitely many more circles are inscribed approaching the corners also as in the sketch. Find the area of the region bounded by the circles.



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Solution by the proposer

Let the circle with radius 1 be "circle 0", and for $1 \le i$, let "circle $i$" be the circle tangent to circle $i-1$ and the top and right-hand sides of the square; and for $0 \le i$ let $r_i$ be the radius of circle $i$. For $1 \le i$ the distance from the center of circle $i-1$ to the top right-hand corner is equal to $\sqrt{2} \cdot r_{i-1}$, and this is also equal to $r_{i-1} + r_i + \sqrt{2} \cdot r_i$. Thus we find the following.

$$\begin{aligned} r_0 &= 1 \\ 1 \le i \;\Rightarrow\; r_{i-1} + r_i + \sqrt{2} \cdot r_i &= \sqrt{2} \cdot r_{i-1}; \\ \text{hence} \quad r_0 &= 1 \quad \text{and} \\ 1 \le i \;\Rightarrow\; r_i &= \frac{\sqrt{2}-1}{\sqrt{2}+1} r_{i-1} \\ &= \frac{(\sqrt{2}-1)^2}{2-1} r_{i-1} \\ &= (3-\sqrt{8}) r_{i-1}. \end{aligned}$$

Thus since $r_0 = 1$ we find by a trivial induction that $0 \le i \;\Rightarrow\; r_i = (3-\sqrt{8})^i$. It now follows that the area $A$ of all the circles is as follows.

$$\begin{aligned} A &= \pi \left[ r_0^2 + 4(r_1^2 + r_2^2 + r_3^2 + \cdots) \right] \\ &= \pi \left[ 1 + 4 \sum_{i=1}^{\infty} [(3-\sqrt{8})^i]^2 \right] \\ &= \pi \left[ 1 + 4 \frac{17 - 6\sqrt{8}}{1 - (17 - 6\sqrt{8})} \right] \\ &= \pi \left[ 1 + 4 \frac{17 - 6\sqrt{8}}{6\sqrt{8} - 16} \right] \\ &= \frac{3\sqrt{2} - 2}{2} \pi \end{aligned}$$

This is approximately 88.0683% of the area of the square.

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Also solved by Lewis Felver, Kourosh Ghaderi, John M. Sayer, and Winston Teitler