Problem for 2012 May

Proposed by Dan Jurca

Suppose $a \in \mathbf{R}$ and $1 < a$, and consider the sequence as follows.

$$\left(1,\ \sqrt{a},\ \sqrt{a\sqrt{a}},\ \sqrt{a\sqrt{a\sqrt{a}}},\ \sqrt{a\sqrt{a\sqrt{a\sqrt{a}}}},\ \ldots\right)$$

1. Show that the sequence converges, and find the limit, L.
2. Show that the sequence converges linearly.

   This means that with $x_0 = 1,\ x_1 = \sqrt{a},\ x_2 = \sqrt{a\sqrt{a}},\ \ldots$ and $\varepsilon_n = L - x_n$

$$\lim_{n\to\infty}\left|\frac{\varepsilon_n}{\varepsilon_{n-1}}\right|$$

   exists and equals a positive real number.

Solution by the proposer

With $x_0 = 1$ and $1 \le n \;\Rightarrow\; x_n = \sqrt{ax_{n-1}}$ we will show that $(x_n)_{n=0}^{\infty} \to a$ and that the limit above (the *asymptotic error constant)* equals one-half. First we show that $(x_n)_{n=0}^{\infty}$ strictly increases. For clearly $1 = x_0 < x_1 = \sqrt{a}$ since $1 < a$; and if $2 \le n$ and $x_{n-2} < x_{n-1}$, then $ax_{n-2} < ax_{n-1}$ so $\sqrt{ax_{n-2}} < \sqrt{ax_{n-1}}$ so that $x_{n-1} < x_n$. Thus (by induction) the sequence increases. Next we show the sequence is bounded (by a). Again clearly $1 = x_0 < a$ by hypothesis; and if $1 \le n$ and $x_{n-1} < a$, then $ax_{n-1} < a^2$, so $x_n = \sqrt{ax_{n-1}} < \sqrt{a^2} = a$. Thus (again by induction) $0 \le n \;\Rightarrow\; x_n < a$. It follows that the increasing and bounded sequence $(x_n)_{n=0}^{\infty}$ converges. But if

$$\begin{aligned}
\lim_{n\to\infty} x_n &= L, \quad \text{then}\\
a\lim_{n\to\infty} x_n &= aL, \quad \text{so}\\
\lim_{n\to\infty} ax_n &= aL, \quad \text{whence}\\
\lim_{n\to\infty} \sqrt{ax_n} &= \sqrt{aL}, \quad \text{so}\\
L &= \sqrt{aL},
\end{aligned}$$

from which $L^2 = aL$, and since $0 < L$ it follows that $L = a$. Therefore $(x_n)_{n=0}^{\infty} \to a$.

Now with $\varepsilon_n = a - x_n$ for $0 \le n$, we have $x_n = a - \varepsilon_n$; hence $\varepsilon_n = a - x_n = a - \sqrt{ax_{n-1}} = a - \sqrt{a(a-\varepsilon_{n-1})}$. Thus $\varepsilon_n - a = -\sqrt{a(a-\varepsilon_{n-1})}$, so that $\varepsilon_n^2 - 2a\varepsilon_n + a^2 = a^2 - a\varepsilon_{n-1}$. Hence we have the following equation. $\varepsilon_n^2 - 2a\varepsilon_n + a\varepsilon_{n-1} = 0$. Solving for ε_n we find as follows.

$$\begin{aligned}
\varepsilon_n &= \frac{2a \pm \sqrt{4a^2 - 4a\varepsilon_{n-1}}}{2}\\
&= a \pm \sqrt{a^2 - a\varepsilon_{n-1}}\\
&= a - \sqrt{a^2 - a\varepsilon_{n-1}} \quad (\text{since } \varepsilon_n < a)\\
&= a - \sqrt{a^2 - a\varepsilon_{n-1}} \times \frac{a + \sqrt{a^2 - a\varepsilon_{n-1}}}{a + \sqrt{a^2 - a\varepsilon_{n-1}}}\\
&= \frac{a\varepsilon_{n-1}}{a + \sqrt{a^2 - a\varepsilon_{n-1}}} \quad \text{so}\\
\frac{\varepsilon_n}{\varepsilon_{n-1}} &= \frac{a}{a + \sqrt{a^2 - a\varepsilon_{n-1}}} \to \frac{1}{2} \text{ as } n \to \infty \ (\text{since } \varepsilon_n \to 0).
\end{aligned}$$

Also solved by Kourosh Ghaderi, Thomas Kim, Massoud Malek, John M. Sayer, and Winston Teitler