**Problem for 2012 November**

Proposed by Dan Jurca

If $0 < r$ and for each positive integer $n$

$$a_n = \sqrt[r]{1} + \sqrt[r]{2} + \sqrt[r]{3} + \cdots + \sqrt[r]{n-1} + \sqrt[r]{n},$$

does the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{a_n}$ converge?

---

Solution by the proposer

Yes, the series converges.

By the mean value theorem for integrals if $1 \leq k$, then there exists $\xi \in (k-1, k)$ such that

$$\int_{k-1}^{k} \sqrt[r]{x}\, dx = \sqrt[r]{\xi}, \quad \text{so that}$$

$$\frac{r}{r+1}\left[k^{(r+1)/r} - (k-1)^{(r+1)/r}\right] < \sqrt[r]{k}. \quad \text{Hence}$$

$$\sum_{k=1}^{n} \int_{k-1}^{k} \sqrt[r]{x}\, dx < \sum_{k=1}^{n} \sqrt[r]{k}, \quad \text{so}$$

$$\frac{r}{r+1} \sum_{k=1}^{n} \left[k^{(r+1)/r} - (k-1)^{(r+1)/r}\right] < a_n, \quad \text{so, finally,}$$

$$\frac{r}{r+1} n^{(r+1)/r} < a_n, \quad \text{whence}$$

$$\frac{1}{a_n} < \frac{r+1}{r} \frac{1}{n^{(1+1/r)}}.$$

Thus by comparison with the "$p$-series" with $p = 1 + 1/r$,

$$\sum_{n=1}^{\infty} \frac{1}{a_n} < \infty.$$

---

Also solved by Maryna Longnickel and John Sayer