Problem for 2012 December

Proposed by Dan Jurca

The following is a slight modification of problem (5) on page 198 of *A Second Course in Linear Algebra* by William C. Brown.

Find a sequence $(x_n)_{n=0}^{\infty}$ such that

1. $0 \leq n \ \Rightarrow \ x_n \in [0,1]$;
2. if $y \in [0,1]$, then there exists a subsequence $(x_{n_k})_{k=0}^{\infty}$ of $(x_n)_{n=0}^{\infty}$ such that
   - $(x_{n_k})_{k=0}^{\infty}$ is a non-decreasing sequence, and
   - the sequence $(x_{n_k})_{k=0}^{\infty}$ converges to y.

Solution by the proposer

Consider the sequence $(x_n)_{n=0}^{\infty}$ with terms as follows.

$$\frac{0}{1}, \ \frac{0}{2}, \frac{1}{2}, \ \frac{0}{3}, \frac{1}{3}, \frac{2}{3}, \ \frac{0}{4}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \ \frac{0}{5}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \ \ldots$$

The terms of this sequence occur in “blocks”, where for $0 \leq i$ the i-th block consists of the $i+1$ nonnegative rationals here.

$$\frac{0}{i+1}, \frac{1}{i+1}, \frac{2}{i+1}, \frac{3}{i+1}, \frac{4}{i+1}, \ldots, \frac{i}{i+1}$$

To compute x_n let $i = \lfloor(\sqrt{8n+1} - 1)/2\rfloor$ and then $x_n = (n - i(i+1)/2)/(i+1)$. (Thus x_n occurs in block i, $0 \leq i$.)

If $0 \leq k \ \Rightarrow \ n_k = k(k+1)/2$, then $x_{n_k} = 0$, so $(x_{n_k})_{k=0}^{\infty}$ is non-decreasing and converges to 0. If $y \in (0,1]$, then let $n_0 = 0$, so that $x_{n_0} = 0$. If $1 \leq k$, choose n_k as follows.

Let $\rho = \min\{y - x_{n_{k-1}}, 1/k\}$, choose a rational $q \in (y - \rho, y)$, and say $q = x_n$ where $n_{k-1} < n$. Since $X = \{x_n \mid 0 \leq n\}$ is dense in $[0,1]$, as is the complement $X - F$ where F is a finite subset of X, such an n exists. Then with $n_k = n$ we find

- $n_{k-1} < n = n_k$, and
- $x_{n_{k-1}} < x_{n_k}$, (for $\rho \leq y - x_{n_{k-1}}$, so $x_{n_{k-1}} \leq y - \rho < q = x_n = x_{n_k}$), and
- $0 < y - x_{n_k} = y - x_n = y - q < \rho \leq 1/k$.

Then clearly $(x_{n_k})_{k=0}^{\infty}$ is non-decreasing and converges to y.