Problem for 2013 February

Proposed by Dan Jurca

Let $f : \mathbf{R} \to \mathbf{R}$ by $f(x) = |x|$; find a sequence $(f_n)_{n=1}^{\infty}$ with the following properties.

- $1 \le n \;\Rightarrow\; f_n : \mathbf{R} \to \mathbf{R}$.
- $1 \le n \;\Rightarrow\; f_n$ is continuously differentiable.
- $1 \le n$ and $x \in \mathbf{R} \;\Rightarrow\; |f_n'(x)| \le 1$.
- $(f_n)_{n=1}^{\infty}$ converges uniformly to f.

Solution by the proposer

We define for each $n,\ 1 \le n$, the function $f_n : \mathbf{R} \to \mathbf{R}$ as follows.

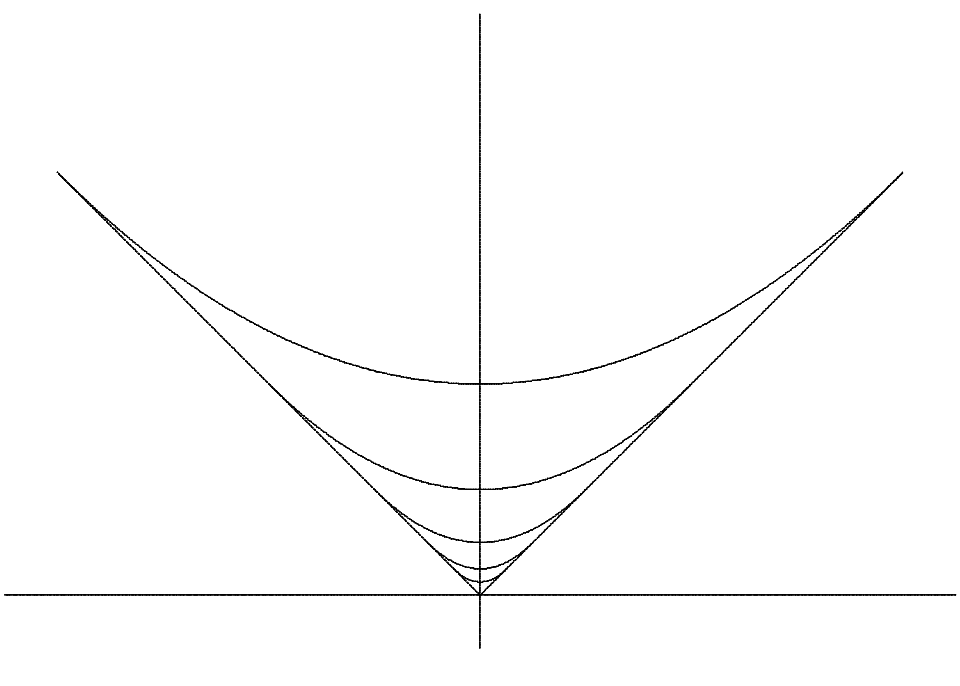
$$f_n(x) = \begin{cases} |x| & \text{if } x \in (-\infty, -1/n] \cup [1/n, \infty) \\ \dfrac{nx^2}{2} + \dfrac{1}{2n} & \text{if } x \in [-1/n, 1/n] \end{cases}$$

$f_n(\pm 1/n) = 1/n$, so each f_n is well-defined and continuous. Then one easily checks that

$$f_n'(x) = \begin{cases} -1 & \text{if } x \in (-\infty, -1/n] \\ nx & \text{if } x \in [-1/n, 1/n] \\ 1 & \text{if } x \in [1/n, \infty) \end{cases},$$

so that each f_n is continuously differentiable, and $x \in \mathbf{R} \;\Rightarrow\; |f_n'(x)| \le 1$. Next, if $x \in (-\infty, -1/n] \cup [1/n, \infty)$, then $f_n(x) = |x| = f(x)$, so $|f(x) - f_n(x)| = 0$; and if $x \in [-1/n, 1/n]$, then $0 \le |x| = f(x) \le 1/n$ and $1/(2n) \le f_n(x) \le 1/n$, so $-1/n \le -f_n(x) \le -1/(2n)$, from which $-1/n \le |x| - f_n(x) \le 1/(2n)$, and $|f(x) - f_n(x)| \le 1/n$. Therefore $x \in \mathbf{R} \;\Rightarrow\; |f(x) - f_n(x)| \le 1/n$, so that $(f_n)_{n=1}^{\infty} \to f$ uniformly.

The following sketch shows graphs of $f_1,\ f_2,\ f_4,\ f_8,\ f_{16}$, and f on the interval $[-1, 1]$.



Also solved by Charles Burnette (Drexel University) and Winston Teitler