Problem for 2013 April

Communicated by Dan Jurca

The following problem (with slightly different notation) appears on page 11 of *Differential Geometry* by James J. Stoker. Here $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, and $\mathbf{d}$ are vectors in $\mathbf{R}^3$.

Verify the identity of Lagrange:

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = \begin{vmatrix} \mathbf{a}\cdot\mathbf{c} & \mathbf{b}\cdot\mathbf{c} \\ \mathbf{a}\cdot\mathbf{d} & \mathbf{b}\cdot\mathbf{d} \end{vmatrix}.$$

Solution 1 by Dan Jurca

If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k},\ \mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k},\ \mathbf{c} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$, and $\mathbf{d} = d_1\mathbf{i} + d_2\mathbf{j} + d_3\mathbf{k}$, with $\mathbf{i}, \mathbf{j}$, and $\mathbf{k}$ the usual orthogonal unit vectors in $\mathbf{R}^3$, we find

$$\begin{aligned}
\mathbf{a}\times\mathbf{b} &= (a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k} \quad \text{and} \\
\mathbf{c}\times\mathbf{d} &= (c_2d_3 - c_3d_2)\mathbf{i} + (c_3d_1 - c_1d_3)\mathbf{j} + (c_1d_2 - c_2d_1)\mathbf{k}, \quad \text{so} \\
(\mathbf{a}\times\mathbf{c})\cdot(\mathbf{c}\times\mathbf{d}) &= a_2b_3c_2d_3 - a_2b_3c_3d_2 - a_3b_2c_2d_3 + a_3b_2c_3d_2 + \\
&\quad a_3b_1c_3d_1 - a_3b_1c_1d_3 - a_1b_3c_3d_1 + a_1b_3c_1d_3 + \\
&\quad a_1b_2c_1d_2 - a_1b_2c_2d_1 - a_2b_1c_1d_2 + a_2b_1c_2d_1; \quad \text{next} \\
(\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) &= (a_1c_1 + a_2c_2 + a_3c_3)(b_1d_1 + b_2d_2 + b_3d_3) \quad \text{and} \\
(\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c}) &= (a_1d_1 + a_2d_2 + a_3d_3)(b_1c_1 + b_2c_2 + b_3c_3), \quad \text{so} \\
(\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c}) &= (a_1b_1c_1d_1 + a_1b_2c_1d_2 + a_1b_3c_1d_3 + a_2b_1c_2d_1 + a_2b_2c_2d_2 + a_2b_3c_2d_3 + \\
&\quad a_3b_1c_3d_1 + a_3b_2c_3d_2 + a_3b_3c_3d_3) - (a_1b_1c_1d_1 + a_1b_2c_2d_1 + a_1b_3c_3d_1 + \\
&\quad a_2b_1c_1d_2 + a_2b_2c_2d_2 + a_2b_3c_3d_2 + a_3b_1c_1d_3 + a_3b_2c_2d_3 + a_3b_3c_3d_3),
\end{aligned}$$

and since the terms with $a_1b_1c_1d_1,\ a_2b_2c_2d_2$, and $a_3b_3c_3d_3$ cancel, the identity holds.

Solution 2 by Dan Jurca

With the identities

$$\begin{aligned}
\mathbf{a}\cdot\mathbf{b} &= \mathbf{b}\cdot\mathbf{a} \\
\mathbf{a}\times\mathbf{b} &= -\mathbf{b}\times\mathbf{a} \\
\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) &= (\mathbf{a}\times\mathbf{b})\cdot\mathbf{c} \\
\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) &= (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c} \quad \text{we find}
\end{aligned}$$

$$\begin{aligned}
(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) &= [(\mathbf{a}\times\mathbf{b})\times\mathbf{c}]\cdot\mathbf{d} \\
&= -[\mathbf{c}\times(\mathbf{a}\times\mathbf{b})]\cdot\mathbf{d} \\
&= -[(\mathbf{c}\cdot\mathbf{b})\mathbf{a} - (\mathbf{c}\cdot\mathbf{a})\mathbf{b}]\cdot\mathbf{d} \\
&= (\mathbf{c}\cdot\mathbf{a})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{c}\cdot\mathbf{b})(\mathbf{a}\cdot\mathbf{d}) \\
&= (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c}) \\
&= \begin{vmatrix} \mathbf{a}\cdot\mathbf{c} & \mathbf{b}\cdot\mathbf{c} \\ \mathbf{a}\cdot\mathbf{d} & \mathbf{b}\cdot\mathbf{d} \end{vmatrix}.
\end{aligned}$$

Also solved by Winston Teitler