Problem for 2013 May

Proposed by Dan Jurca

Generalizing such formulas as $\cos 2\theta = \cos^2\theta - \sin^2\theta$ and $\sin 3\theta = 3\sin\theta\cos^2\theta - \sin^3\theta$, show that if $s = \sin\theta$ and $c = \cos\theta$, then for each positive integer n

$$\sin n\theta = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} (-1)^k \binom{n}{2k+1} c^{n-(2k+1)} s^{2k+1} \qquad \text{and}$$

$$\cos n\theta = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n}{2k} c^{n-2k} s^{2k}.$$

Solution by the proposer

From Euler's identity $e^{i\theta} = \cos\theta + i\sin\theta$ we find the following.

$$\begin{aligned}
\cos n\theta + i\sin n\theta &= e^{i(n\theta)} \\
&= e^{n(i\theta)} \\
&= (e^{i\theta})^n \\
&= (\cos\theta + i\sin\theta)^n \\
&= (c + is)^n \\
&= \sum_{k=0}^{n} \binom{n}{k} c^{n-k} (is)^k \\
&= \sum_{k=0}^{n} i^k \binom{n}{k} c^{n-k} s^k
\end{aligned}$$

and the result follows from the fact that $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, $\ldots$, and then by equating real and imaginary parts.

Also solved by Massoud Malek and Winston Teitler