Problem for 2013 July

Proposed by Dan Jurca

The following problem is similar to one given in an examination at the University of Illinois.

> Suppose $0 < a < b$ and A is the annulus $A = \{z \in \mathbf{C} \mid a < |z| < b\}$. Does there exist a sequence $(f_n)_{n=0}^{\infty}$ of functions analytic in $\mathbf{C}$ such that $(f_n)_{n=0}^{\infty}$ converges uniformly in A to the function $f : A \to \mathbf{C}$ by $f(z) = 1/z$?

Solution by the proposer

No, there exists no such sequence. We assume otherwise and obtain a contradiction. So suppose each f_n is analytic in $\mathbf{C}$, and $(f_n)_{n=0}^{\infty}$ converges uniformly in A to f. Then with $\varepsilon = 1/(2b)$, there exists a natural number N such that $N \le n \;\Rightarrow\; |f(z) - f_n(z)| < \varepsilon$ for each $z \in A$. Hence $N \le n$ and $z \in A \;\Rightarrow\; |1/z - f_n(z)| < \varepsilon$, so that

$$\begin{aligned} z \in A \;\Rightarrow\; |zf_N(z) - 1| &< \varepsilon|z| \\ &= 1/(2b) \cdot |z| \\ &= 1/2 \cdot |z|/b \\ &< 1/2. \end{aligned}$$

Since the function $z \mapsto zf_N(z) - 1$ is an analytic function, it follows from the maximum modulus principle that $|zf_N(z) - 1| < 1$ in the disk $\{z \in \mathbf{C} \mid |z| < b\}$. In particular with $z = 0$ we have

$$|0 \cdot f_N(0) - 1| = |0 - 1| = |-1| = 1 < 1,$$

an obvious contradiction.