Problem for 2013 August

Proposed by Dan Jurca

This problem is suggested by one which the proposer saw in a magazine.

Find a matrix A such that $(I+A)^{-1} = I + A^{-1}$.

Solution by the proposer

$$\text{If} \quad A = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$$

$$\text{then} \quad A^{-1} = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\text{and} \quad I + A = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$$

$$\begin{aligned} \text{so} \quad (I+A)^{-1} &= \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}; \quad \textit{i.e.}, \end{aligned}$$

$$(I+A)^{-1} = I + A^{-1}.$$

Two more examples:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & -1 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & -1 \end{bmatrix}$$

It is clear that if A is $n \times n$, $(I+A)^{-1} = I + A^{-1}$, and P is an invertible $n \times n$ matrix, then

$$\begin{aligned} (I + P^{-1}AP)^{-1} &= [P^{-1}(I+A)P]^{-1} \\ &= P^{-1}(I+A)^{-1}P \\ &= P^{-1}(I+A^{-1})P \\ &= P^{-1}IP + P^{-1}A^{-1}P \\ &= I + (P^{-1}AP)^{-1}. \end{aligned}$$

Also solved by Maryna Longnickel, Valerie Taigos, and Jan van Delden

The solution by Valerie Taigos shows that each A such that $(I+A)^{-1} = I + A^{-1}$ is similar to one such as above.