Problem for 2013 September

Proposed by Dan Jurca

By a *nice triangle* we mean a triangle T such that

i. the length of each side of T is a positive integer, and
ii. the area of T equals the perimeter of T.

a. Find a nice triangle which is a right triangle.
b. Find a nice triangle which is not a right triangle.
c. Does there exist a nice triangle which is an isosceles triangle?

Solution by the proposer

a. If T is a nice triangle and is a right triangle with altitude a, base b, and hypotenuse $\sqrt{a^2+b^2}$, then

$$\begin{aligned} \frac{ab}{2} &= a+b+\sqrt{a^2+b^2} \quad \text{so} \\ a^2+b^2 &= \left(\frac{ab}{2}-a-b\right)^2 \\ &= \frac{a^2b^2}{4}+a^2+b^2-a^2b-ab^2+2ab, \quad \text{so} \\ (a^2-4a)b^2-(4a^2-8a)b &= 0, \end{aligned}$$

and since $a \neq 0$ and $b \neq 0$, it follows that $b = (4a-8)/(a-4)$. Then if $a = 5$, it follows that $b = 12$; and the area and perimeter of the (pythagorean) right triangle with sides of lengths 5, 12, and 13 equal 30.

b. Suppose T is a triangle with sides of lengths a, b, and c, and area A; then with $p = a+b+c$ (by Heron's formula)

$$\begin{aligned} \text{if} \quad & p = A \\ \text{then} \quad & p = \sqrt{\frac{p}{2}\left(\frac{p}{2}-a\right)\left(\frac{p}{2}-b\right)\left(\frac{p}{2}-c\right)} \quad \text{so} \\ (-a+b+c)(a-b+c)(a+b-c) &= 16(a+b+c). \end{aligned}$$

A simple computer search for triples $a \le b \le c$ of positive integers for which this equation holds yields, for example, $a = 9$, $b = 10$, and $c = 17$; the area and the perimeter of the triangle with sides of these lengths equal 36; obviously $9^2+10^2 = 181 \neq 289 = 17^2$, so this is not a right triangle.

c. There does not exist a nice triangle which is also an isosceles triangle. For suppose T is such a triangle with altitude of length a, base length b, and two sides of length $c = \sqrt{a^2+(b/2)^2}$. Then $a = \sqrt{c^2-b^2/4}$, so equating area and perimeter we find

$$\begin{aligned} \frac{b\sqrt{c^2-b^2/4}}{2} &= b+2c \quad \text{so} \\ b\sqrt{4c^2-b^2} &= 4b+8c \quad \text{from which} \\ 4b^2c^2-b^4 &= 64c^2+64bc+16b^2 \quad \text{and} \\ (4b^2-64)c^2-64bc-(b^4+16b^2) &= 0. \quad \text{Solving for } c \\ c &= \frac{64b \pm \sqrt{64^2b^2+4(4b^2-64)(b^4+16b^2)}}{2(4b^2-64)} \\ &= \frac{64b \pm \sqrt{4096b^2+16b^2(b^2-16)(b^2+16)}}{8(b^2-16)} \\ &= \frac{64b \pm 4b\sqrt{256+b^4-256}}{8(b^2-16)} \\ &= \frac{16b \pm b^3}{2(b^2-16)}. \end{aligned}$$

Now if

$$c = \frac{16b - b^3}{2(b^2-16)}, \quad \text{then}$$
$$c = b\frac{16-b^2}{2(b^2-16)} = -\frac{b}{2},$$

which is not possible since b and c are positive integers. Hence

$$c = \frac{b^3+16b}{2(b^2-16)}.$$

If b is an odd integer, then the numerator of this fraction is odd but the denominator is even, so c is not an integer; therefore b must be an even integer. By checking cases we find that this expression does not evaluate to a positive integer if $0 \le b < 4$ or $4 < b \le 17$. Next,

$$\begin{aligned} 17 < b \ \Rightarrow\ 9 < b - 8 \quad &\text{so} \\ 9^2 < (b-8)^2 \quad &\text{so} \\ 80 < b^2 - 16b + 64 \quad &\text{and} \\ 16 < b^2 - 16b \quad &\text{so} \\ 0 < 16b < b^2 - 16 \quad &\text{whence} \\ 0 < \frac{16b}{b^2-16} < 1, & \end{aligned}$$

and since

$$\frac{b^3+16b}{2(b^2-16)} = \frac{b}{2} + \frac{16b}{b^2-16},$$

the expression above for c does not evaluate to a positive integer if b is a positive integer. Hence no isosceles triangle is a nice triangle.

(One may also let $f : \mathbf{R} - \{-4, 4\} \to \mathbf{R}$ by $f(x) = 16x/(x^2-16)$, find $f'(x) = -(16x^2+256)/(x^2-16)^2 < 0$, deduce that $4 < x \ \Rightarrow\ 0 < f(x)$, f strictly decreases in $(4,\infty)$, and observe that $f(17) = 272/273 < 1$. Thus, again, $17 \le x \ \Rightarrow\ 16x/(x^2-16)$ is not an integer.)

Remark. The altitude of the isosceles triangle with length of base $b = 12$ and two sides of length $c = 15/2$ is of length $9/2$; the area and perimeter of this triangle equal 27, so this isosceles triangle is "almost nice". The area of the isosceles triangle with base length 6 and two sides of length 5 equals 12, and the perimeter equals 16; the proposer has found no isosceles triangle with integral lengths of sides and an area and perimeter (both integers) differing by less than 4. The perimeter of the equilateral triangle with sides of length 7 equals 21, and the area equals $49\sqrt{3}/4 = 21.217622...$.

Also solved by Valerie Taigos