Problem for 2013 October

Proposed by Dan Jurca

Suppose $a \in \mathbf{R}$ and $f : [-1, 1] \to \mathbf{R}$ by $f(x) = \sin(a \sin^{-1} x)$; show that f is the restriction (to $[-1, 1]$) of a polynomial function if and only if a is 0 or an odd integer.

Solution by the proposer

If $f(x) = y = \sin(a \sin^{-1} x)$, then $\sin^{-1} y = a \sin^{-1} x$; differentiating (twice) we find

$$\begin{aligned} \frac{y'}{\sqrt{1-y^2}} &= \frac{a}{\sqrt{1-x^2}} \\ (1-x^2)(y')^2 &= a^2(1-y^2) \\ -2x \cdot (y')^2 + (1-x^2) \cdot 2y'y'' &= a^2 \cdot -2yy' \\ -xy' + (1-x^2)y'' &= -a^2 y, \end{aligned}$$

so we have the differential equation $(1-x^2)y'' - xy' + a^2 y = 0$. Since f is an analytic function near 0,

$$y = \sum_{n=0}^{\infty} c_n x^n \quad \text{where} \quad c_n = \frac{f^{(n)}(0)}{n!}.$$

We find the c_n's using a standard technique. Thus

$$y = \sum_{n=0}^{\infty} c_n x^n \quad \text{so} \quad a^2 y = \sum_{n=0}^{\infty} a^2 c_n x^n \tag{1}$$

$$y' = \sum_{n=0}^{\infty} (n+1) c_{n+1} x^n \quad \text{so} \quad xy' = \sum_{n=0}^{\infty} n c_n x^n \tag{2}$$

$$y'' = \sum_{n=0}^{\infty} (n+1)(n+2) c_{n+2} x^n \quad \text{so} \quad x^2 y'' = \sum_{n=0}^{\infty} (n-1) n c_n x^n \quad \text{and}$$

$$(1-x^2)y'' = \sum_{n=0}^{\infty} [(n+1)(n+2)c_{n+2} - (n-1)nc_n] x^n, \tag{3}$$

and combining (1), (2), and (3) we have for $n = 0,\ 1,\ 2,\ 3,\ \ldots$

$$(n+1)(n+2)c_{n+2} - (n-1)nc_n - nc_n + a^2 c_n = 0.$$

Since, obviously $c_0 = f(0) = 0$, and $c_1 = f'(0) = a$, we have

$$c_0 = 0, \quad c_1 = a, \quad \text{and } 2 \le n \;\Rightarrow\; c_n = \frac{(n-2)^2 - a^2}{(n-1)n} c_{n-2}.$$

Hence

$$\begin{aligned} c_3 &= (1^2 - a^2)/(2 \cdot 3) \cdot a = a \cdot (1^2 - a^2)/(3 \cdot 2) \\ c_5 &= (3^2 - a^2)/(4 \cdot 5) \cdot a \cdot (1^2 - a^2)/(2 \cdot 3) = a \cdot (1^2 - a^2)(3^2 - a^2)/(5 \cdot 4 \cdot 3 \cdot 2) \\ c_7 &= (5^2 - a^2)/(6 \cdot 7) \cdot a \cdot (1^2 - a^2)(3^2 - a^2)/(5 \cdot 4 \cdot 3 \cdot 2), \quad \text{and in general} \end{aligned}$$

$$c_n = \frac{a}{n!} \cdot \begin{cases} 0 & \text{if } n \text{ is even} \\ \displaystyle\prod_{i=1}^{\lfloor n/2 \rfloor} [(2i-1)^2 - a^2] & \text{if } n \text{ is odd.} \end{cases}$$

If a is not 0 or an odd integer, then $c_n \neq 0$ for infinitely many n, so f is not a polynomial function; if a is 0 or an odd integer, then $c_n \neq 0$ for only finitely many n, so f is a polynomial function.

Also solved by Valerie Taigos