Problem for 2013 November

Communicated by Dan Jurca

Prove that if

- n is a positive integer, and k is an integer greater than 1; and
- $p(x)$ is a polynomial of degree n in $\mathbf{C}[x]$; and
- $a \in \mathbf{C}$; and
- $A = (a_{ij})$ is the $(n+k)\times(n+k)$ matrix where $1 \le i \le n+k,\ 1 \le j \le n+k \ \Rightarrow\ a_{ij} = p(a+i+j)$; then

0 is an eigenvalue of A.

Solution by Dan Jurca

Lemma. If $1 \le n,\ p(x)$ is a polynomial of degree n in $\mathbf{C}[x]$, and $q(x) = p(x+1) - p(x)$, then the degree of $q(x)$ is $n-1$.

Proof.

If $n = 1$, then $p(x) = c_1 x + c_0$ for some $c_1, c_0 \in \mathbf{C},\ c_1 \ne 0$; hence

$$q(x) = p(x+1) - p(x) = [c_1(x+1) + c_0] - [c_1 x + c_0] = c_1,$$

a non-zero constant polynomial, so the assertion of the lemma holds if $n = 1$. Suppose now that $2 \le n$, the assertion of the lemma holds for each $m,\ m < n$, and the degree of $p(x)$ is n. Then $p(x) = c_n x^n + r(x)$ where $c_n \ne 0$ and the degree of $r(x)$ is less than n (or $r(x) = 0$). Then

$$\begin{aligned} q(x) &= p(x+1) - p(x) \\ &= [c_n(x+1)^n + r(x+1)] - [c_n x^n + r(x)] \\ &= \left[c_n x^n + c_n \sum_{j=1}^{n} \binom{n}{j} x^{n-j} + \ r(x+1) \right] - [c_n x^n + r(x)] \\ &= n c_n x^{n-1} + c_n \sum_{j=2}^{n} \binom{n}{j} x^{n-j} \ + \ r(x+1) - r(x), \end{aligned}$$

a polynomial of degree $n-1$, so the lemma follows by induction on n.

Now if the $(n+k)\times(n+k)$ matrix of polynomials $A[x]$ is defined by $A_{ij}[x] = p(x+i+j)$, where $p(x)$ is the polynomial in the problem statement above, then $A[x]$ is singular. For if, for $i = n+k-1,\ n+k-2,\ \ldots,\ 1$, row i is subtracted from row $i+1$, then in the resulting matrix each row below row 1 consists, by the lemma, of polynomials of degree $n-1$. Repeating, now with $i = n+k-1,\ n+k-2,\ \ldots,\ 2$, and subtracting row i from row $i+1$, each row below row 2 consists of polynomials of degree $n-2$. Repeating $n-2$ more times (if $3 \le n$) results in a matrix with at least two rows equal, so the original matrix is row-equivalent to a singular matrix; hence 0 is an eigenvalue of A.

Here is an example with $n = k = 2,\ p(x) = 2x^2 + 3x + 4$, and $A_{ij}[x] = p(x+i+j)$.

$$A[x] = \begin{bmatrix} 2x^2+11x+18 & 2x^2+15x+31 & 2x^2+19x+\ \ 48 & 2x^2+23x+\ \ 69 \\ 2x^2+15x+31 & 2x^2+19x+48 & 2x^2+23x+\ \ 69 & 2x^2+27x+\ \ 94 \\ 2x^2+19x+48 & 2x^2+23x+69 & 2x^2+27x+\ \ 94 & 2x^2+31x+123 \\ 2x^2+23x+69 & 2x^2+27x+94 & 2x^2+31x+123 & 2x^2+35x+156 \end{bmatrix}$$

$$\to \begin{bmatrix} 2x^2+11x+18 & 2x^2+15x+31 & 2x^2+19x+48 & 2x^2+23x+69 \\ 4x+13 & 4x+17 & 4x+21 & 4x+25 \\ 4x+17 & 4x+21 & 4x+25 & 4x+29 \\ 4x+21 & 4x+25 & 4x+29 & 4x+33 \end{bmatrix}$$

$$\to \begin{bmatrix} 2x^2+11x+18 & 2x^2+15x+31 & 2x^2+19x+48 & 2x^2+23x+69 \\ 4x+13 & 4x+17 & 4x+21 & 4x+25 \\ 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 \end{bmatrix}$$