Problem for 2014 January

Proposed by Dan Jurca

A sketch of the graph of the function "tan" suggests, and one may easily prove, that for each integer n there exists a unique θ_n such that $(n-1/2)\pi < \theta_n < (n+1/2)\pi$ and $\tan\theta_n = \theta_n$.
(Thus each interval $((n-1/2)\pi, (n+1/2)\pi)$ contains precisely one fixed point, θ_n, of tan.)
Prove the following.

$$1 \le n \;\Rightarrow\; (n+1/2)\pi - \frac{1}{(n+9/20)\pi} < \theta_n < (n+1/2)\pi - \frac{1}{(n+1/2)\pi}$$

Solution by the proposer

For $n = 1,\ 2,\ \ldots$ let $\alpha_n = (n+1/2)\pi - 1/((n+9/20)\pi)$ and $\beta_n = (n+1/2)\pi - 1/((n+1/2)\pi)$; we show $\tan\alpha_n < \alpha_n$ and $\beta_n < \tan\beta_n$. (The motivation here is the observation that $\cot z = 1/z - z/3 - \cdots$ for z near—but different from—zero.)

Proposition 1. $1 \le n \;\Rightarrow\; (n+9/20)\pi - \dfrac{1}{3(n+9/20)\pi} < \alpha_n = (n+1/2)\pi - \dfrac{1}{(n+9/20)\pi}$.

Proof.

Since $800 < 87\pi^2$ it follows that $800 - 27\pi^2 < 60\pi^2$; therefore

$$\begin{aligned}
\frac{800-27\pi^2}{60\pi^2} &< 1; \quad \text{hence} \\
1 \le n \;\Rightarrow\; \frac{40}{3\pi^2} - \frac{9}{20} &< n \quad \text{so} \\
\frac{40}{3\pi^2} &< n + 9/20 \quad \text{so} \\
\frac{2}{3(n+9/20)\pi} &< \frac{\pi}{20} \quad \text{so} \\
\frac{1}{(n+9/20)\pi} - \frac{1}{3(n+9/20)\pi} &< \frac{10\pi}{20} - \frac{9\pi}{20} \quad \text{so} \\
\frac{9}{20}\pi - \frac{1}{3(n+9/20)\pi} &< \frac{1}{2}\pi - \frac{1}{(n+9/20)\pi} \quad \text{so} \\
(n+9/20)\pi - \frac{1}{3(n+9/20)\pi} &< (n+1/2)\pi - \frac{1}{(n+9/20)\pi} \\
&= \alpha_n.
\end{aligned}$$

Proposition 2. If $\psi : [0,\pi] \to \mathbf{R}$ by $\psi(t) = \sin t - t\cos t$, then $0 \le \psi$; and $0 < t \le \pi \;\Rightarrow\; 0 < \psi(t)$.

Proof.

$\psi'(t) = \cos t - \cos t + t\sin t = t\sin t$, so $t \in (0,\pi) \;\Rightarrow\; 0 < \psi'(t)$, and ψ strictly increases; since $\psi(0) = 0$, the result follows, and $t \in (0,\pi] \;\Rightarrow\; 0 < \psi(t)$.

Proposition 3. If $\varphi : [0,\pi] \to \mathbf{R}$ by $\varphi(t) = (3-t^2)\sin t - 3t\cos t$, then $0 \le \varphi$.

Proof.

$\varphi'(t) = -2t\sin t + (3-t^2)\cos t - 3\cos t + 3t\sin t = t\sin t - t^2\cos t = t(\sin t - t\cos t) = t\psi(t)$, and since $\varphi(0) = 0$, the result follows from proposition 2; moreover, $0 < t \le \pi \;\Rightarrow\; 0 < \varphi(t)$.

Corollary. $\theta \in (0,\pi) \;\Rightarrow\; \cot\theta < 1/\theta - \theta/3$.

Proof.

By proposition 3: $0 < \theta < \pi \;\Rightarrow\; 0 < 3\sin\theta - \theta^2\sin\theta - 3\theta\cos\theta \;\Rightarrow\; 3\theta\cos\theta < 3\sin\theta - \theta^2\sin\theta$; therefore (dividing by the positive quantity $3\theta\sin\theta$)

$$\begin{aligned}
\frac{\cos\theta}{\sin\theta} &< \frac{1}{\theta} - \frac{\theta}{3}, \quad \text{and} \\
0 < \theta < \pi \;\Rightarrow\; \cot\theta &< \frac{1}{\theta} - \frac{\theta}{3}.
\end{aligned}$$

Proposition 4. $1 \le n$ and $\alpha_n = (n+1/2)\pi - \dfrac{1}{(n+9/20)\pi} \;\Rightarrow\; \tan\alpha_n < \alpha_n$.

Proof.

$$\begin{aligned}
\tan\alpha_n &= \tan\left[\left(n+\frac{1}{2}\right)\pi - \frac{1}{(n+9/20)\pi}\right] \\
&= \tan\left[n\pi + \left(\frac{\pi}{2} - \frac{1}{(n+9/20)\pi}\right)\right] \\
&= \tan\left(\frac{\pi}{2} - \frac{1}{(n+9/20)\pi}\right) \quad \text{by periodicity of } \tan \\
&= \cot\left(\frac{1}{(n+9/20)\pi}\right) \\
&< (n+9/20)\pi - \frac{1}{3(n+9/20)\pi} \quad \text{by the corollary} \\
&< \alpha_n \quad \text{by proposition 1,}
\end{aligned}$$

proving proposition 4.

Proposition 5. If $\varphi : [0,\pi/2] \to \mathbf{R}$ by $\varphi(t) = t\cos t + t^2\sin t - \sin t$, then $0 \le \varphi$; and $0 < t \le \pi/2 \;\Rightarrow\; 0 < \varphi(t)$.

Proof.

$\varphi'(t) = \cos t - t\sin t + 2t\sin t + t^2\cos t - \cos t = t\sin t + t^2\cos t$, so $t \in (0,\pi/2] \;\Rightarrow\; 0 < \varphi'(t)$, and φ strictly increases; since $\varphi(0) = 0$, the result follows. Moreover, $t \in (0,\pi/2] \;\Rightarrow\; 0 < \varphi(t)$.

Proposition 6. $\theta \in (0,\pi/2) \;\Rightarrow\; 1/\theta - \theta < \cot\theta$.

Proof.

By proposition 5

$$\begin{aligned}
\theta \in (0,\pi/2) \;\Rightarrow\; 0 < \theta\cos\theta + \theta^2\sin\theta - \sin\theta \quad &\text{so} \\
\sin\theta - \theta^2\sin\theta < \theta\cos\theta \quad &\text{so dividing by the positive } \theta\sin\theta \\
\frac{1}{\theta} - \theta < \frac{\cos\theta}{\sin\theta} \quad &\text{and} \\
\frac{1}{\theta} - \theta < \cot\theta.&
\end{aligned}$$

Proposition 7. $1 \le n$ and $\beta_n = (n+1/2)\pi - \dfrac{1}{(n+1/2)\pi} \;\Rightarrow\; \beta_n < \tan\beta_n$.

Proof.

By proposition 6 with $\theta = \dfrac{1}{(n+1/2)\pi} \in (0,\pi/2)$

$$\begin{aligned}
(n+1/2)\pi - \frac{1}{(n+1/2)\pi} &< \cot\left[\frac{1}{(n+1/2)\pi}\right] \quad \text{so} \\
\beta_n &< \tan\left[\frac{\pi}{2} - \frac{1}{(n+1/2)\pi}\right] \\
&= \tan\left[n\pi + \left(\frac{\pi}{2} - \frac{1}{(n+1/2)\pi}\right)\right] \\
&= \tan\left[(n+1/2)\pi - \frac{1}{(n+1/2)\pi}\right] \\
&= \tan\beta_n.
\end{aligned}$$

By propositions 4 and 7 we now have $1 \le n \;\Rightarrow\; \tan\alpha_n - \alpha_n < 0 < \tan\beta_n - \beta_n$, and it follows that $\alpha_n < \theta_n < \beta_n$.