Problem for 2014 February

Communicated by Dan Jurca

Prove that if k is an integer greater than 1, then for each integer $a \in \mathbf{Z}$ there exist integers m and n such that $a^k = n^2 - m^2$.

Solution by Dan Jurca

If $a^{k-2} = n^2 - m^2$, then $a^k = a^2 \cdot a^{k-2} = a^2 \cdot (n^2 - m^2) = a^2n^2 - a^2m^2 = (an)^2 - (am)^2$; and of course $a^2 = a^2 - 0^2$. Therefore by induction (on k) it is sufficient to show that for each integer a there exist integers m and n such that $a^3 = n^2 - m^2$.

If a is an integer, then one of $a-1,\ a$ is even, and similarly one of $a,\ a+1$ is even; therefore for each integer a each of $(a-1)a/2$ and $a(a+1)/2$ is an integer. Then

$$\begin{aligned} a^3 &= a \cdot a^2 \\ &= \left[\left(\frac{a^2+a}{2}\right) - \left(\frac{a^2-a}{2}\right)\right] \cdot \left[\left(\frac{a^2+a}{2}\right) + \left(\frac{a^2-a}{2}\right)\right] \\ &= \left(\frac{a^2+a}{2}\right)^2 - \left(\frac{a^2-a}{2}\right)^2, \end{aligned}$$

and a^3 equals the difference of two squares.

We can express the result as follows. If a is an integer, k is an integer and $2 \le k$, then

$$a^k = \begin{cases} (a^{k/2})^2 - 0^2 & \text{if } k \text{ is even} \\ \left[a^{(k-3)/2} \cdot \dfrac{a^2+a}{2}\right]^2 - \left[a^{(k-3)/2} \cdot \dfrac{a^2-a}{2}\right]^2 & \text{if } k \text{ is odd,} \end{cases}$$

so that a^k equals the difference of two squares.

Also solved by Swaarnabja Bhaumik, Arthur Fabian, Marina Longnickel, and Massoud Malek.