

**Problem for 2014 May**

Communicated by Dan Jurca

The following is an instance of a well-known problem.

For  $1 \leq n$  and  $0 \leq r$  let  $B(n, r)$  be the (closed)  $n$ -dimensional ball of radius  $r$  centered at the origin; *i.e.*,

$$\begin{aligned} B(n, r) &= \{\mathbf{x} \in \mathbf{R}^n \mid \|\mathbf{x}\|_2 \leq r\} \\ &= \{(x_1, x_2, \dots, x_n) \in \mathbf{R}^n \mid \sqrt{x_1^2 + x_2^2 + \dots + x_n^2} \leq r\}. \end{aligned}$$

A point  $(x_1, x_2, \dots, x_n) \in \mathbf{R}^n$  is an integer lattice point if each coordinate  $x_i$  is an integer; *i.e.*, each  $x_i \in \mathbf{Z}$ . Let  $L(n, r) = |\mathbf{Z}^n \cap B(n, r)|$ ; so  $L(n, r)$  equals the number of integer lattice points in the  $n$ -dimensional ball of radius  $r$  centered at the origin. For example,  $L(2, 1) = 5$ ,  $L(2, 3) = 13$ , and  $L(3, 3) = 123$ .

Compute  $L(4, 5)$ .

Solution by Dan Jurca

Clearly  $L(1, r) = 1 + 2\lfloor r \rfloor$ . If  $2 \leq n$ , one can picture the  $n$ -ball of radius  $r$  centered at the origin as “slices” of  $(n - 1)$ -balls, and deduce that

$$2 \leq n \Rightarrow L(n, r) = L(n - 1, r) + 2 \sum_{i=1}^{\lfloor r \rfloor} L(n - 1, \sqrt{r^2 - i^2}).$$

Thus one can recursively compute  $L(n, r)$  for each  $n$  and  $r$ . However, if one uses a digital computer to perform the computations, one finds rounding errors resulting from approximations of the square roots, so direct application of the formulas above is unreliable. Next one may consider all  $n$ -tuples  $(x_1, x_2, \dots, x_n) \in \mathbf{Z}^n$  where  $1 \leq i \leq n \Rightarrow |x_i| \leq r$  and count for how many of these  $n$ -tuples it is true that  $x_1^2 + x_2^2 + \dots + x_n^2 \leq r^2$ , but for large  $n$  and  $r$  this may be inefficient. (There are  $(2\lfloor r \rfloor + 1)^n$   $n$ -tuples to consider.)

So we define (for  $0 \leq n$  and  $0 \leq x$ )  $Q(n, x) = L(n, \sqrt{x})$ . Then we have  $L(n, r) = Q(n, r^2)$ , and

$$\begin{aligned} Q(1, x) &= L(1, \sqrt{x}) = 1 + 2\lfloor \sqrt{x} \rfloor \quad \text{and} \\ 2 \leq n &\Rightarrow Q(n, x) = L(n, \sqrt{x}) = Q(n - 1, x) + 2 \sum_{i=1}^{\lfloor \sqrt{x} \rfloor} Q(n - 1, x - i^2). \end{aligned}$$

For nonnegative integer  $r$  one can compute  $\lfloor \sqrt{r} \rfloor$  exactly (and efficiently). “Unrolling” the recursion above one finds, for example,

$$\begin{aligned} L(4, 5) = Q(4, 25) &= Q(1, 25) \\ &+ 6 \times Q(1, 24) \\ &+ 12 \times Q(1, 23) \\ &+ 8 \times Q(1, 22) \\ &+ 6 \times Q(1, 21) \\ &+ 24 \times Q(1, 20) \\ &+ 24 \times Q(1, 19) \\ &+ 12 \times Q(1, 17) \\ &+ 30 \times Q(1, 16) \\ &+ 24 \times Q(1, 15) \\ &+ 24 \times Q(1, 14) \\ &+ 8 \times Q(1, 13) \\ &+ 24 \times Q(1, 12) \\ &+ 48 \times Q(1, 11) \\ &+ 6 \times Q(1, 9) \\ &+ 48 \times Q(1, 8) \\ &+ 36 \times Q(1, 7) \\ &+ 24 \times Q(1, 6) \\ &+ 24 \times Q(1, 5) \\ &+ 48 \times Q(1, 4) \\ &+ 24 \times Q(1, 3) \\ &+ 24 \times Q(1, 1) \\ &+ 30 \times Q(1, 0) = 3121. \end{aligned}$$