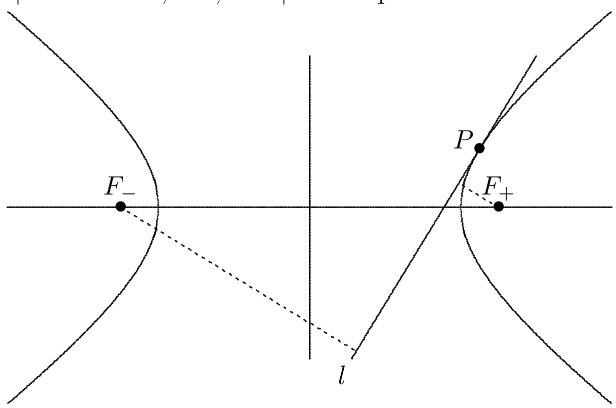
Problem for 2014 July

Proposed by Dan Jurca

This is similar to problem 1.1.6 (ii) of *Elementary Differential Geometry* (2^{nd} ed.) by Andrew Pressley.

Consider a hyperbola with foci at F_- and F_+. At point P in the hyperbola the line l tangent to the hyperbola is drawn, d_- is the distance from F_- to l, and d_+ is the distance from F_+ to l.
Prove that the product d_-d_+ is constant; *i.e.*, d_-d_+ is independent of P.



Solution by the proposer

Suppose an equation for the hyperbola is $x^2/a^2 - y^2/b^2 = 1$. Then the foci are at $\pm(c = \sqrt{a^2+b^2})$, and differentiating (implicitly) $x/a^2 - yy'/b^2 = 0$. If the point P is at (u, v) where $v \neq 0$, then the slope of line l is $m = b^2u/(a^2v)$, so an equation of l is $y - v = m(x - u)$, or $b^2ux - a^2vy + a^2v^2 - b^2u^2 = 0$, and since $u^2/a^2 - v^2/b^2 = 1$, we have (for l) $b^2ux - a^2vy - a^2b^2 = 0$.
Hence the distances d_- from F_- (at $(-c, 0)$) to l and d_+ from F_+ (at $(c, 0)$) to l are, respectively,

$$d_- = \frac{|b^2u \cdot -\sqrt{a^2+b^2} - a^2b^2|}{\sqrt{(b^2u)^2 + (a^2v)^2}} \qquad d_+ = \frac{|b^2u \cdot \sqrt{a^2+b^2} - a^2b^2|}{\sqrt{(b^2u)^2 + (a^2v)^2}}.$$

Therefore

$$d_-d_+ = \frac{|b^4u^2(a^2+b^2) - a^4b^4|}{b^4u^2 + a^4v^2}.$$

But since $u^2/a^2 - v^2/b^2 = 1$, we have $b^2u^2 - a^2v^2 = a^2b^2$, so $a^2b^4u^2 - a^4b^2v^2 = a^4b^4$, and it follows that $a^2b^4u^2 + b^6u^2 - a^4b^4 = b^6u^2 + a^4b^2v^2$, which is nonnegative. Therefore

$$|b^4u^2(a^2+b^2) - a^4b^4| = b^4u^2(a^2+b^2) - a^4b^4 \qquad \text{so that}$$

$$\begin{aligned} d_-d_+ &= \frac{b^4u^2(a^2+b^2) - a^4b^4}{b^4u^2 + a^2(a^2v^2)} \\ &= \frac{a^2b^4u^2 + b^6u^2 - a^4b^4}{b^4u^2 + a^2(b^2u^2 - a^2b^2)} \\ &= \frac{b^2(a^2b^2u^2 + b^4u^2 - a^4b^2)}{b^4u^2 + a^2b^2u^2 - a^4b^2} \\ &= \frac{b^2(a^2b^2u^2 + b^4u^2 - a^4b^2)}{a^2b^2u^2 + b^4u^2 - a^4b^2} \\ &= b^2, \end{aligned}$$

independently of P. Finally, if $v = 0$, then (l is parallel to the y-axis)

$$d_-d_+ = \begin{cases} (c - |a|)(c + |a|) = c^2 - a^2 = b^2 & \text{if } u = -|a| \\ (c + |a|)(c - |a|) = c^2 - a^2 = b^2 & \text{if } u = +|a|, \end{cases}$$

again independently of P.

Also solved by Marina Longnickel