Problem for 2014 August

Proposed by Dan Jurca

One easily shows that if $0 < a < 1/e$, (where e is the base of natural logarithms), then there exist exactly two real solutions of the following equation.

$$\frac{\ln x}{x} = a$$

For each real $x,\ e < x$, let $z(x)$ be the greater of the two real solutions of the equation where $a = 1/x$; thus $e < z(x)$ and

$$\frac{\ln z(x)}{z(x)} = \frac{1}{x}.$$

Prove that $z(x) \sim x \ln x$; *i.e.*,

$$\lim_{x\to\infty} \frac{z(x)}{x \ln x} = 1.$$

Solution by the proposer

We show that

$$e^e \le x \ \Rightarrow \ x\ln x < z(x) < \left(1 + 2\frac{\ln\ln x}{\ln x}\right) \cdot x \ln x,$$

from which the limit follows at once by the "squeeze theorem". (A tighter upper bound on $z(x)$ is possible, but this is sufficient.)

Suppose $e < x$, and let $f : (e, \infty) \to \mathbf{R}$ by

$$e < t \ \Rightarrow \ f(t) = \frac{1}{x} - \frac{\ln t}{t}.$$

Then

$$\lim_{e^+} f = \frac{1}{x} - \frac{1}{e} < 0 < \frac{1}{x} = \lim_{\infty} f, \text{ and } \frac{df}{dt}(t) = -\frac{1 - \ln t}{t^2} = \frac{\ln t - 1}{t^2} > 0,$$

so that f strictly increases, and there exists a unique $z(x),\ e < z(x)$, such that $f(z(x)) = 0$. Since

$$\begin{aligned} f(x\ln x) &= \frac{1}{x} - \frac{\ln(x\ln x)}{x\ln x} \\ &= \frac{1}{x}\left(1 - \frac{\ln x + \ln\ln x}{\ln x}\right) \\ &= \frac{1}{x}\left(1 - 1 - \frac{\ln\ln x}{\ln x}\right) \\ &= -\frac{1}{x} \cdot \frac{\ln\ln x}{\ln x} \\ &< 0, \end{aligned}$$

it follows that $x \ln x < z(x)$.

To obtain an upper bound we use the following

Proposition. $e^e \le x \;\Rightarrow\; 1 + 2\dfrac{\ln\ln x}{\ln x} < \ln x.$

Proof.

If $\varphi : [e, \infty) \to \mathbf{R}$ by $\varphi(x) = \ln x - 1 - 2\dfrac{\ln\ln x}{\ln x}$, then

$$\begin{aligned}
\varphi'(x) &= \frac{1}{x} - 2 \cdot \frac{(1/\ln x \cdot 1/x) \cdot \ln x - \ln\ln x \cdot 1/x}{(\ln x)^2} \\
&= \frac{1}{x} \cdot \left[1 - 2 \cdot \frac{1 - \ln\ln x}{(\ln x)^2}\right] \\
&= \frac{1}{x} \cdot \frac{(\ln x)^2 + 2(\ln\ln x - 1)}{(\ln x)^2}, \quad \text{so}
\end{aligned}$$

$$e^e \le x \;\Rightarrow\; \varphi'(x) > 0.$$

Therefore, since φ strictly increases in $[e^e, \infty)$ and $0 < \varphi(e^e) = e - 1 - 2/e$, we find that $e^e \le x \;\Rightarrow\; 0 < \varphi(x)$, and the proposition is proved.

Now if $e^e \le x$, then

$$\begin{aligned}
1 + 2\frac{\ln\ln x}{\ln x} &< \ln x \qquad \text{so} \\
\ln\left(1 + 2\frac{\ln\ln x}{\ln x}\right) &< \ln\ln x \qquad \text{so} \\
\ln\left(1 + 2\frac{\ln\ln x}{\ln x}\right) + \ln\ln x &< 2\ln\ln x \qquad \text{so} \\
\ln\left(1 + 2\frac{\ln\ln x}{\ln x}\right) + \ln x + \ln\ln x &< \ln x + 2\ln\ln x \\
&= \left(1 + 2\frac{\ln\ln x}{\ln x}\right)\ln x \qquad \text{so} \\
\frac{\ln\left[\left(1 + 2\frac{\ln\ln x}{\ln x}\right) x\ln x\right]}{\left(1 + 2\frac{\ln\ln x}{\ln x}\right)\ln x} &< 1 \qquad \text{so} \\
\frac{\ln\left[\left(1 + 2\frac{\ln\ln x}{\ln x}\right) x\ln x\right]}{\left(1 + 2\frac{\ln\ln x}{\ln x}\right) x\ln x} &< \frac{1}{x} \qquad \text{so}
\end{aligned}$$

$$\begin{aligned}
0 &< \frac{1}{x} - \frac{\ln\left[\left(1 + 2\frac{\ln\ln x}{\ln x}\right) x\ln x\right]}{\left(1 + 2\frac{\ln\ln x}{\ln x}\right) x\ln x} \\
&= f\left[\left(1 + 2\frac{\ln\ln x}{\ln x}\right) x\ln x\right].
\end{aligned}$$

It follows that $e^e \le x \;\Rightarrow\; x\ln x < z(x) < \left(1 + 2\dfrac{\ln\ln x}{\ln x}\right) x\ln x$, and therefore

$$15.154262\ldots = e^e < x \;\Rightarrow\; 1 < \frac{z(x)}{x\ln x} < 1 + 2\frac{\ln\ln x}{\ln x},$$

so $z(x) \sim x\ln x$.