Problem for 2,014 September

Proposed by Dan Jurca

Evaluate the following.

$$\sqrt{1{,}014{,}172 - 2{,}014\sqrt{123}} \; + \; \sqrt{1{,}014{,}172 + 2{,}014\sqrt{123}}$$

Solution by the proposer

More generally suppose $a,\ b$, and c are nonnegative and $0 \le a - b\sqrt{c}$; let $u = \sqrt{a - b\sqrt{c}}$ and $v = \sqrt{a + b\sqrt{c}}$. Then

$$\begin{aligned} u^2 &= a - b\sqrt{c} \\ uv &= \sqrt{a^2 - b^2 c} \\ v^2 &= a + b\sqrt{c} \qquad \text{so} \\ (u+v)^2 &= u^2 + 2uv + v^2 \\ &= (a - b\sqrt{c}) + 2\sqrt{a^2 - b^2 c} + (a + b\sqrt{c}) \\ &= 2a + 2\sqrt{a^2 - b^2 c}. \end{aligned}$$

Therefore $u + v = \sqrt{a - b\sqrt{c}} + \sqrt{a + b\sqrt{c}} = \sqrt{2a + 2\sqrt{a^2 - b^2 c}}$.

Now if $a = 1{,}014{,}172,\ b = 2{,}014$, and $c = 123$, then

$$\begin{aligned} \sqrt{a - b\sqrt{c}} + \sqrt{a + b\sqrt{c}} &= \sqrt{2 \times 1{,}014{,}172 + 2\sqrt{1{,}014{,}172^2 - 2014^2 \times 123}} \\ &= \sqrt{2{,}028{,}344 + 2\sqrt{1{,}028{,}544{,}845{,}584 - 4{,}056{,}196 \times 123}} \\ &= \sqrt{2{,}028{,}344 + 2\sqrt{1{,}028{,}544{,}845{,}584 - 498{,}912{,}108}} \\ &= \sqrt{2{,}028{,}344 + 2\sqrt{1{,}028{,}045{,}933{,}476}} \\ &= \sqrt{2{,}028{,}344 + 2 \times 1{,}013{,}926} \\ &= \sqrt{2{,}028{,}344 + 2{,}027{,}852} \\ &= \sqrt{4{,}056{,}196} \\ &= 2{,}014. \end{aligned}$$

Also solved by Jan van Delden (the Netherlands) and Winston Teitler