Problem for 2015 January

Proposed by Dan Jurca

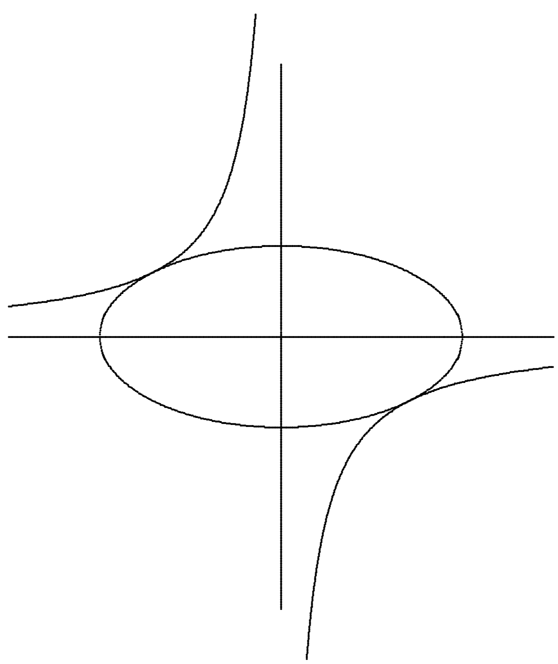
Determine all points (x,y) in the Cartesian x-y plane such that

$$x^3y + 4xy^3 + x^2 - 4xy + 4y^2 = 4.$$

Solution by the proposer

$$\begin{aligned}
x^3y + 4xy^3 + x^2 - 4xy + 4y^2 = 4 \quad &\Rightarrow \\
x^3y + 4xy^3 + x^2 - 4xy + 4y^2 - 4 = 0 \quad &\Rightarrow \\
x^3y + 4xy^3 - 4xy + x^2 + 4y^2 - 4 = 0 \quad &\Rightarrow \\
(xy+1)(x^2 + 4y^2 - 4) = 0 \quad &\Rightarrow \\
(xy+1)\left(\frac{x^2}{4} + \frac{y^2}{1} - 1\right) = 0.&
\end{aligned}$$

Therefore if $x^3y+4xy^3+x^2-4xy+4y^2 = 4$, then either $xy+1 = 0$ or $x^2/4+y^2/1-1 = 0$; so the points (x,y) such that the given equation holds lie on either the hyperbola with equation $y = -1/x$ or the ellipse with equation $x^2/4 + y^2/1 = 1$. Conversely, if a point with coordinates (x,y) lies on either the hyperbola with equation $y = -1/x$ or on the ellipse with equation $x^2/4 + y^2/1 = 1$, then $xy + 1 = 0$ or $x^2/4 + y^2/1 - 1 = 0$, so $x^3y + 4xy^3 + x^2 - 4xy + 4y^2 = 4$. The set of points in question is therefore the union of these two curves.



Also solved by John M. Sayer and Winston Teitler

There were two incomplete solutions.