Problem for 2015 February

Proposed by Dan Jurca

The following is a slight variation of a problem which appeared on a recent Putnam exam.

Suppose $f : \mathbf{R} \to \mathbf{R}$ is differentiable and for each $x \in \mathbf{R}$

$$\begin{aligned} f'(x) &= \frac{f(x+1) - f(x)}{1} \\ &= \frac{f(x+2) - f(x)}{2}. \end{aligned}$$

Find f.

Solution by the proposer

Lemma. With f as above: if $x \in \mathbf{R}$, then $f'(x+1) = f'(x)$.

Proof.

$$\begin{aligned} f'(x+1) &= \frac{f((x+1)+1) - f(x+1)}{1} \\ &= f(x+2) - f(x+1) \\ \text{and} \quad f'(x) &= \frac{f(x+1) - f(x)}{1} \\ &= f(x+1) - f(x) \\ \text{so} \quad f'(x+1) + f'(x) &= [f(x+2) - f(x+1)] + [f(x+1) - f(x)] \\ &= f(x+2) - f(x) \\ &= 2 \cdot \frac{f(x+2) - f(x)}{2} \\ &= 2 \cdot f'(x), \\ \text{whence} \quad f'(x+1) &= f'(x), \text{ proving the lemma.} \end{aligned}$$

Now suppose $h \in \mathbf{R}$. Then for each $x \in \mathbf{R}$

$$\begin{aligned} f'(x+h) - f'(x) &= \frac{f((x+h)+1) - f(x+h)}{1} - \frac{f(x+1) - f(x)}{1} \\ &= [f((x+1)+h) - f(x+h)] - [f(x+1) - f(x)] \\ &= [f((x+1)+h) - f(x+1)] - [f(x+h) - f(x)] \end{aligned}$$

so if $h \neq 0$, then

$$\frac{f'(x+h) - f'(x)}{h} = \frac{f((x+1)+h) - f(x+1)}{h} - \frac{f(x+h) - f(x)}{h}$$

and by differentiability of f at $x+1$ and at x

$$\begin{aligned} \lim_{h \to 0} \frac{f'(x+h) - f'(x)}{h} &= \lim_{h \to 0} \frac{f((x+1)+h) - f(x+1)}{h} - \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= f'(x+1) - f'(x) \\ &= 0 \quad \text{by the lemma,} \end{aligned}$$

so that $f''(x) = 0$ for each $x \in \mathbf{R}$; therefore $f(x) = f'(0) \cdot x + f(0)$, a linear (more precisely, *affine*) function.

Also solved by Charles Burnette (graduate student, Drexel University), Jan van Delden (the Netherlands), and Winston Teitler