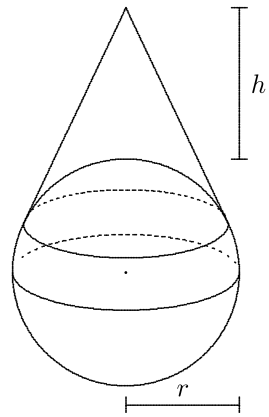
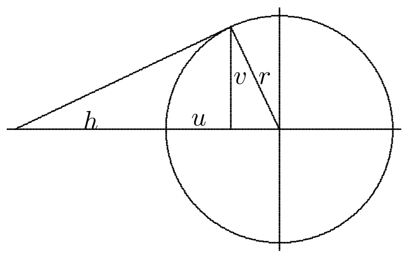
Problem for 2015 March

Proposed by Dan Jurca

What fraction of the surface area of a sphere of radius r is visible from a point at distance h above the surface of the sphere?



Solution by the proposer



The visible area equals that of a surface of revolution as follows.

$$SA = 2\pi \int_{-r}^{-r+u} y\sqrt{1 + (y')^2}\, dx$$

where $x^2 + y^2 = r^2$. Since then $2x + 2yy' = 0$, we have $y' = -x/y$, and $\sqrt{1 + (y')^2} = \sqrt{1 + x^2/y^2} = \sqrt{x^2 + y^2}/y = r/y$. Hence

$$SA = 2\pi \int_{-r}^{-r+u} y \cdot \frac{r}{y}\, dx = 2\pi \int_{-r}^{-r+u} r\, dx = 2\pi ru.$$

From the similar triangles in the sketch we find $(r - u)/v = v/(h + u)$, so $v^2 = hr - hu + ru - u^2$, and since $(r - u)^2 + v^2 = r^2$, we also have $v^2 = 2ru - u^2$. Therefore $hr - hu + ru - u^2 = 2ru - u^2$, so $u = hr/(h + r)$. It follows that the fraction of the surface area of the sphere is as follows.

$$\begin{aligned} \frac{SA}{4\pi r^2} &= \frac{2\pi r \cdot hr/(h + r)}{4\pi r^2} \\ &= \frac{1}{2}\frac{h}{h + r} \end{aligned}$$

If we let $h = \rho r$, this can be written as

$$\frac{1}{2}\frac{\rho}{\rho + 1},$$

which, as expected, equals 0 when $\rho = h = 0$, and approaches $1/2$ as $\rho \to \infty$.

Also solved by Charles Burnette (graduate student, Drexel University), Marina Longnickel, John M. Sayer, and Winston Teitler