Problem for 2015 June

Proposed by Dan Jurca

By a *diagonal* in a polygon P we mean a line segment from one vertex of P to a different vertex of P. Let a_n equal the mean (average) length of the diagonals in a regular n-gon inscribed in the unit circle. (Examples: $a_3 = \sqrt{3}$; $a_4 = (4\sqrt{2}+4)/6$.) Find

$$\lim_{n\to\infty} a_n.$$

Solution by the proposer

The limit in question equals $4/\pi$.

Proof.

Labeling the n vertices $V_0,\ V_1,\ \ldots,\ V_{n-1}$ of a regular n-gon such that the coordinates of V_j equal

$$\left(\cos\frac{2\pi j}{n}, \sin\frac{2\pi j}{n}\right),\ j = 0,\ 1,\ \ldots,\ n-1,$$

for $j = 1,\ 2,\ \ldots,\ n-1$ the length d_j of the diagonal from V_0 to V_j equals

$$\sqrt{(1-\cos(2\pi j/n))^2 + (0-\sin(2\pi j/n))^2} = \sqrt{2-2\cos(2\pi j/n)} = \sqrt{2}\cdot\sqrt{1-\cos(2\pi j/n)}.$$

Thus the sum of the lengths of diagonals from V_0 to the other $n-1$ vertices equals

$$\sum_{j=1}^{n-1} d_j = \sqrt{2}\cdot\sum_{j=1}^{n-1}\sqrt{1-\cos(2\pi j/n)},$$

so (counting each diagonal only once, not twice) the mean length of all the $n(n-1)/2$ diagonals equals

$$a_n = \frac{1}{2}\cdot n\cdot\frac{\sqrt{2}\cdot\sum_{j=1}^{n-1}\sqrt{1-\cos(2\pi j/n)}}{n(n-1)/2} = \sqrt{2}\cdot\frac{\sum_{j=1}^{n-1}\sqrt{1-\cos(2\pi j/n)}}{n-1} = \sqrt{2}\cdot\frac{n}{n-1}\cdot\frac{\sum_{j=1}^{n}\sqrt{1-\cos(2\pi j/n)}}{n}.$$

By definition of the definite integral as a limit of Riemann sums we have

$$\int_0^{2\pi}\sqrt{1-\cos x}\,dx = \lim_{n\to\infty}\sum_{j=1}^{n}\frac{2\pi}{n}\sqrt{1-\cos\left(j\cdot\frac{2\pi}{n}\right)}, \qquad \text{so}$$

$$\lim_{n\to\infty}\frac{\sum_{j=1}^{n}\sqrt{1-\cos(2\pi j/n)}}{n} = \frac{1}{2\pi}\int_0^{2\pi}\sqrt{1-\cos x}\,dx;$$

$$\begin{aligned}\text{therefore}\quad \lim_{n\to\infty} a_n &= \sqrt{2}\cdot\lim_{n\to\infty}\frac{\sum_{j=1}^{n}\sqrt{1-\cos(2\pi j/n)}}{n}\\ &= \sqrt{2}\cdot\frac{1}{2\pi}\int_0^{2\pi}\sqrt{1-\cos x}\,dx\\ &= \sqrt{2}\cdot\frac{1}{2\pi}\cdot 4\sqrt{2}\\ &= 4/\pi = 1.2732395447\ldots,\end{aligned}$$

as asserted.

Remark. Clearly the mean length of just the interior diagonals also equals $4/\pi$.

Also solved by Marina Longnickel, Massoud Malek, John M. Sayer, and Winston Teitler