Problem for 2015 July

Communicated by Dan Jurca

The following problem appears on page 10 of *The Theory of Groups, An Introduction* by Joseph J. Rotman.

Let G be the additive group of all polynomials in x with coefficients in $\mathbf{Z}$, and let H be the multiplicative group of positive rationals. Prove that $G \cong H$. (*Hint*: Use the fundamental theorem of arithmetic to construct an isomorphism.)

Solution by Dan Jurca

For $i = 0,\ 1,\ 2,\ 3,\ \ldots$, let p_i equal the $(i+1)$-th prime: $p_0 = 2,\ p_1 = 3,\ p_2 = 5,\ p_3 = 7,$ *etc.* (This differs from the more common notation $p_1 = 2,\ p_2 = 3,$ *etc.*) Also let $\mathbf{N}^+ = \{n \in \mathbf{Z} \mid 0 < n\}$ and $\mathbf{Q}^+ = \{q \in \mathbf{Q} \mid 0 < q\}$.

For each prime p there exists the function $\nu_p : \mathbf{N}^+ \to \mathbf{N}$ by

$$n \in \mathbf{N}^+ \ \Rightarrow\ n = \prod\{p_i^{\nu_{p_i}(n)} \mid 0 \le i\} = 2^{\nu_2(n)}3^{\nu_3(n)}5^{\nu_5(n)}7^{\nu_7(n)}\cdots\ ;$$

i.e., $\nu_p(n)$ equals the exponent on p in the expression of n as a product of prime powers. (Of course $\nu_p(1) = 0$ for each p; and, for each n, $\nu_p(n) = 0$ except for finitely many p.)

Suppose $f(x) \in \mathbf{Z}[x]$, so there exist $c_0,\ c_1,\ c_2,\ \ldots,\ c_k \in \mathbf{Z}$ such that $f(x) = c_0 + c_1x + c_2x^2 + \cdots + c_kx^k$, and let $\varphi : \mathbf{Z}[x] \to \mathbf{Q}^+$ as follows.

$$\begin{aligned}\varphi(f[x]) &= \prod\{p_i^{c_i} \mid 0 \le i \le k\}\\ &= 2^{c_0}3^{c_1}5^{c_2}7^{c_3}\cdots p_k^{c_k}\end{aligned}$$

Obviously φ is well-defined. We show that $\varphi : G \cong H$.

Suppose $f(x) \in G = \mathbf{Z}[x]$ and $g(x) \in G$; and we suppose $f(x) = c_0 + c_1x + c_2x^2 + \cdots c_kx^k$ and $g(x) = d_0 + d_1x + d_2x^2 + \cdots + d_lx^l$, where $k \le l$. Then $f(x) = c_0 + c_1x + c_2x^2 + \cdots + c_kx^k + c_{k+1}x^{k+1} + \cdots + c_lx^l$ where $c_{k+1} = c_{k+2} = \cdots = c_l = 0$ (if $k < l$), and $f(x) + g(x) = (c_0 + d_0) + (c_1 + d_1)x + \cdots + (c_l + d_l)x^l$, so

$$\begin{aligned}\varphi[f(x) + g(x)] &= 2^{c_0+d_0}3^{c_1+c_2}5^{c_2+d_2}\cdots p_l^{c_l+d_l}\\ &= (2^{c_0}3^{c_1}5^{c_2}\cdots p_l^{c_l}) \cdot (2^{d_0}3^{d_1}5^{d_2}\cdots p_l^{d_l})\\ &= \varphi[f(x)] \cdot \varphi[g(x)],\end{aligned}$$

so that φ is a homomorphism of groups.

Suppose $q \in \mathbf{Q}^+$; then there exist $m \in \mathbf{N}^+$ and $n \in \mathbf{N}^+$ such that $\gcd(m, n) = 1$, (*i.e.*, m and n have no common prime divisor), and $q = m/n$. Then

$$\begin{aligned}q &= \frac{2^{\nu_2(m)}3^{\nu_3(m)}5^{\nu_5(m)}\cdots}{2^{\nu_2(n)}3^{\nu_3(n)}5^{\nu_5(n)}\cdots}\\ &= 2^{c_0}3^{c_1}5^{c_2}\cdots,\end{aligned}$$

where $c_i = \nu_{p_i}(m) - \nu_{p_i}(n)$ for $i = 0,\ 1,\ 2,\ \ldots$. Clearly the c_i are uniquely specified by q. We define $\psi : \mathbf{Q}^+ \to \mathbf{Z}[x]$ by $\psi(q) = c_0 + c_1x + c_2x^2 + \cdots$. We have at once that $\psi \circ \varphi = \mathrm{id}_G$ and $\varphi \circ \psi = \mathrm{id}_H$, so that φ is a bijective homomorphism, and is therefore an isomorphism.

Also solved by Massoud Malek, John M. Sayer, and Winston Teitler