Problem for 2015 August

Proposed by Dan Jurca

For $n = 1,\ 2,\ 3,\ \ldots$ let

$$H_n = \sum_{i=1}^{n} \frac{1}{i} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

$$X_n = \sum_{i=1}^{n} H_i = H_1 + H_2 + H_3 + \cdots + H_n \quad \text{and}$$

$$Y_n = \sum_{i=1}^{n} X_i = X_1 + X_2 + X_3 + \cdots + X_n.$$

Find formulas in terms of n and H_n for X_n and Y_n.

Solution by the proposer

The following array

$$\begin{aligned}
H_1 &= 1 \\
H_2 &= 1 + 1/2 \\
H_3 &= 1 + 1/2 + 1/3 \\
\vdots &= \quad \vdots \\
H_n &= 1 + 1/2 + 1/3 + \cdots + 1/n
\end{aligned}$$

(when summed along "columns") shows that

$$\begin{aligned}
X_n &= H_1 + H_2 + H_3 + \cdots + H_n \\
&= (n-0)\cdot 1 + (n-1)\cdot 1/2 + (n-2)\cdot 1/3 + (n-3)\cdot 1/4 + \cdots + [n-(n-1)]\cdot 1/n \\
&= n\cdot(1 + 1/2 + 1/3 + \cdots + 1/n) - [0 + 1/2 + 2/3 + 3/4 + \cdots + (n-1)/n] \\
&= nH_n - [(1-1) + (1-1/2) + (1-1/3) + (1-1/4) + \cdots + (1-1/n)] \\
&= nH_n - [n - H_n] \\
&= (n+1)H_n - n,
\end{aligned}$$

and one easily proves by induction on n that this formula holds for each $n,\ 1 \le n$.

Next,

$$\begin{aligned}
Y_n &= \sum_{i=1}^{n} X_i \\
&= \sum_{i=1}^{n} [(i+1)H_i - i] \\
&= \sum_{i=1}^{n} iH_i + \sum_{i=1}^{n} H_i - \sum_{i=1}^{n} i \\
&= \sum_{i=1}^{n} iH_i + (n+1)H_n - n - \frac{n(n+1)}{2}.
\end{aligned}$$

To evaluate the first sum in the above expression we consider the following array.

$$\begin{aligned}
1 \cdot H_1 &= 1 \\
2 \cdot H_2 &= 2 + 2/2 \\
3 \cdot H_3 &= 3 + 3/2 + 3/3 \\
4 \cdot H_4 &= 4 + 4/2 + 4/3 + 4/4 \\
\vdots \;\; &= \;\; \vdots \\
n \cdot H_n &= n + n/2 + n/3 + \cdots + n/n,
\end{aligned}$$

which shows that

$$\begin{aligned}
\sum_{i=1}^{n} iH_i &= \sum_{i=1}^{n} \left[\sum_{j=i}^{n} j \right] \frac{1}{i} \\
&= \sum_{i=1}^{n} \left[\sum_{j=1}^{n} j - \sum_{j=1}^{i-1} j \right] \frac{1}{i} \\
&= \sum_{i=1}^{n} \left[\frac{n(n+1)}{2} - \frac{(i-1)i}{2} \right] \frac{1}{i} \\
&= \frac{n(n+1)}{2} H_n - \frac{1}{2} \sum_{i=1}^{n} (i-1) \\
&= \frac{n(n+1)}{2} H_n - \frac{(n-1)n}{4}.
\end{aligned}$$

Therefore

$$\begin{aligned}
Y_n &= \frac{n(n+1)}{2} H_n - \frac{(n-1)n}{4} + (n+1)H_n - n - \frac{n(n+1)}{2} \\
&= \frac{n^2 + n + 2n + 2}{2} H_n - \frac{n^2 - n + 4n + 2n^2 + 2n}{4} \\
&= \frac{n^2 + 3n + 2}{2} H_n - \frac{3n^2 + 5n}{4} \\
&= \frac{(n+1)(n+2)}{2} H_n - \frac{3n^2 + 5n}{4},
\end{aligned}$$

and one easily proves by induction on n that this formula holds for each $n,\ 1 \leq n$.

One sees that this final expression involves addition, subtraction, multiplication, division, and the numbers 1, 2, 3, 4, and 5.

Also solved by Massoud Malek and John M. Sayer