**Problem for 2015 September and October**

Proposed by Dan Jurca

A drill with radius $r$, $r \le 1/\sqrt{2}$, bores a hole from one corner of a solid cube with edge 1 through the center of the cube to (and through) the diagonally opposite corner, the axis of the drill coinciding with the diagonal of the cube; how much material does the drill remove from the cube?

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Solution by the proposer

Consider first a ("unit") cube in Euclidean 3-space with vertices $P_0, \ldots, P_7$ as follows.

$$P_0 = (0,0,0),\ P_1 = (1,0,0),\ P_2 = (0,1,0),\ P_3 = (1,1,0)$$
$$P_4 = (0,0,1),\ P_5 = (1,0,1),\ P_6 = (0,1,1),\ P_7 = (1,1,1)$$

Rotating first through $\pi/4$ about the $z$-axis, and then through $\cos^{-1}(1/\sqrt{3})$ about the $x$-axis repositions the cube so that the origin $P_0$ remains fixed, and the vertex $P_7$ moves to $(0,0,\sqrt{3})$. We accomplish the first of these motions with the linear transformation sending the standard basis vectors as follows,

$$\langle 1,0,0\rangle \mapsto \langle 1/\sqrt{2}, 1/\sqrt{2}, 0\rangle \qquad \langle 0,1,0\rangle \mapsto \langle -1/\sqrt{2}, 1/\sqrt{2}, 0\rangle \qquad \langle 0,0,1\rangle \mapsto \langle 0,0,1\rangle,$$

and represent this transformation with the following matrix.

$$A_1 = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This sends the point $P_7$ to the point $(0, \sqrt{2}, 1)$. Then the dot product $\langle 0,0,1\rangle \cdot \langle 0,\sqrt{2},1\rangle = 1 = \sqrt{3}\cos\theta$ where $\cos\theta = 1/\sqrt{3}$ and $\sin\theta = \sqrt{2}/\sqrt{3}$. Therefore we accomplish the second of these motions with the linear transformation sending the standard basis vectors as follows,

$$\langle 1,0,0\rangle \mapsto \langle 1,0,0\rangle \qquad \langle 0,1,0\rangle \mapsto \langle 0, 1/\sqrt{3}, \sqrt{2}/\sqrt{3}\rangle \qquad \langle 0,0,1\rangle \mapsto \langle 0, -\sqrt{2}/\sqrt{3}, 1/\sqrt{3}\rangle,$$

and represent this transformation with the following matrix.

$$A_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{3} & -\sqrt{2}/\sqrt{3} \\ 0 & \sqrt{2}/\sqrt{3} & 1/\sqrt{3} \end{bmatrix}$$

Therefore we represent the two motions with the product $A = A_2 A_1$, where $A$ equals the following orthogonal matrix.

$$A = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{6} & 1/\sqrt{6} & -\sqrt{2}/\sqrt{3} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix}$$

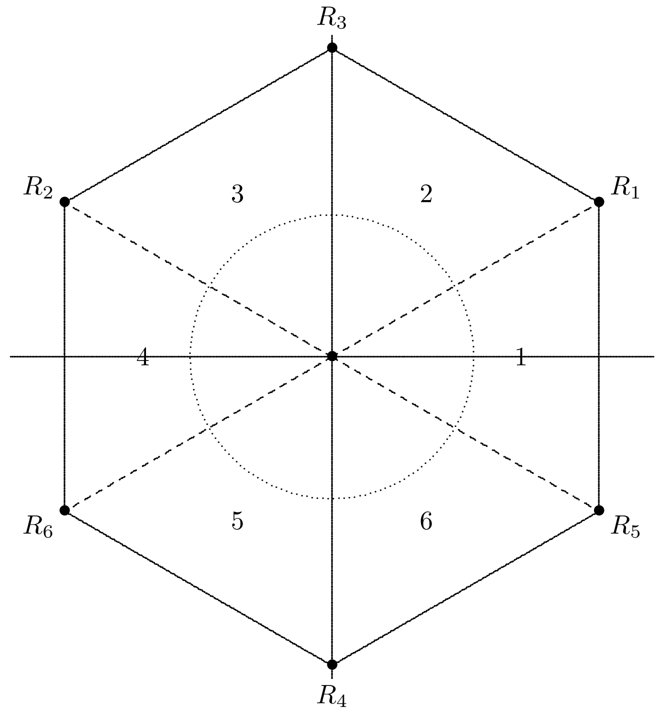
Thus we obtain a (unit) cube with vertices $Q_0, \ldots, Q_7$ as follows. (Here $0 \le i \le 7 \ \Rightarrow\ A : P_i \mapsto Q_i$.)

$$Q_0 = (0,0,0),\ Q_1 = (1/\sqrt{2}, 1/\sqrt{6}, 1/\sqrt{3}),\ Q_2 = (-1/\sqrt{2}, 1/\sqrt{6}, 1/\sqrt{3}),\ Q_3 = (0, \sqrt{2}/\sqrt{3}, 2/\sqrt{3})$$
$$Q_4 = (0, -\sqrt{2}/\sqrt{3}, 1/\sqrt{3}),\ Q_5 = (1/\sqrt{2}, -1/\sqrt{6}, 2/\sqrt{3}),\ Q_6 = (-1/\sqrt{2}, -1/\sqrt{6}, 2/\sqrt{3}),\ Q_7 = (0,0,\sqrt{3})$$

The volume in question then equals the volume of the region common to this rotated cube and the cylinder with equation $x^2 + y^2 = r^2$. To compute this volume we consider the following points $R_0, \ldots, R_7$ obtained by projecting the points $Q_0, \ldots, Q_7$, respectively, onto the $x$-$y$ plane.

$$R_0 = (0,0,0),\ R_1 = (1/\sqrt{2}, 1/\sqrt{6}, 0),\ R_2 = (-1/\sqrt{2}, 1/\sqrt{6}, 0),\ R_3 = (0, \sqrt{2}/\sqrt{3}, 0)$$
$$R_4 = (0, -\sqrt{2}/\sqrt{3}, 0),\ R_5 = (1/\sqrt{2}, -1/\sqrt{6}, 0),\ R_6 = (-1/\sqrt{2}, -1/\sqrt{6}, 0),\ R_7 = (0,0,0)$$

The following sketch shows these points, and six sections $1, 2, \ldots, 6$, each in the shape of an equilateral triangle, determined by them. (We look down from the positive $z$-axis onto the $x$-$y$ plane, with the $x$-axis pointing to the right and the $y$-axis upward.)




Equations of the dashed lines : $y = \pm x/\sqrt{3}$, and of the dotted circle: $x^2 + y^2 = r^2$. The distance from the origin to the line containing $R_1$ and $R_5$ equals $1/\sqrt{2}$; (hence the constraint $r \leq 1/\sqrt{2}$). The dashed lines and the dotted circle intersect in the four points $(\sqrt{3}/2\,r, 1/2\,r)$, $(-\sqrt{3}/2\,r, 1/2\,r)$, $(-\sqrt{3}/2\,r, -1/2\,r)$, and $(\sqrt{3}/2\,r, -1/2\,r)$.

The volume of the region in question equals the sum of the volumes of six regions — that above section 1 in the cylinder, that above section 2 in the cylinder, . . ., and that above section 6 in the cylinder. We observe that six planes bound the rotated cube — three containing the point $Q_0$, and three containing the point $Q_7$. The volume of the region above section 1 equals that within the cylinder with equation $x^2 + y^2 = r^2$, below the plane containing points $Q_1, Q_3, Q_5$, and $Q_7$, and above the plane containing the points $Q_0, Q_1, Q_4$, and $Q_5$. (We observe that the upper plane contains the point $Q_7$ and must also contain $Q_1$ and $Q_5$, since these points project onto $R_1$ and $R_5$, respectively, and only the plane $Q_1Q_3Q_5Q_7$ contains $Q_7$ and also $Q_1$ and $Q_5$; and the lower plane contains the point $Q_0$ and must also contain the points $Q_1$ and $Q_5$.) We reason similarly for the other five regions, and summarize in the following table.

| region | section | above plane | below plane |
|---|---|---|---|
| 1 | $R_0R_1R_5$ | $Q_0Q_1Q_4Q_5$ | $Q_1Q_3Q_5Q_7$ |
| 2 | $R_0R_1R_3$ | $Q_0Q_1Q_2Q_3$ | $Q_1Q_3Q_5Q_7$ |
| 3 | $R_0R_2R_3$ | $Q_0Q_1Q_2Q_3$ | $Q_2Q_3Q_6Q_7$ |
| 4 | $R_0R_2R_6$ | $Q_0Q_2Q_4Q_6$ | $Q_2Q_3Q_6Q_7$ |
| 5 | $R_0R_4R_6$ | $Q_0Q_2Q_4Q_6$ | $Q_4Q_5Q_6Q_7$ |
| 6 | $R_0R_4R_5$ | $Q_0Q_1Q_4Q_5$ | $Q_4Q_5Q_6Q_7$ |

Next we find equations for the six planes which bound the rotated cube, and display them here.

| plane | equation |
|---|---|
| $Q_0Q_1Q_2Q_3$ | $z = \sqrt{2}\,y$ |
| $Q_0Q_1Q_4Q_5$ | $z = \sqrt{3}/\sqrt{2}\,x - 1/\sqrt{2}\,y$ |
| $Q_0Q_2Q_4Q_6$ | $z = -\sqrt{3}/\sqrt{2}\,x - 1/\sqrt{2}\,y$ |
| $Q_1Q_3Q_5Q_7$ | $z = -\sqrt{3}/\sqrt{2}\,x - 1/\sqrt{2}\,y + \sqrt{3}$ |
| $Q_2Q_3Q_6Q_7$ | $z = \sqrt{3}/\sqrt{2}\,x - 1/\sqrt{2}\,y + \sqrt{3}$ |
| $Q_4Q_5Q_6Q_7$ | $z = \sqrt{2}\,y + \sqrt{3}$ |

Next we express the volume of the region in question as the sum of six triple (iterated) integrals as follows.

$$\begin{aligned}
V(r) = {} & \int_{-r/2}^{r/2} \int_{\sqrt{3}|y|}^{\sqrt{r^2-y^2}} \int_{\sqrt{3}/\sqrt{2}\,x-1/\sqrt{2}\,y}^{-\sqrt{3}/\sqrt{2}\,x-1/\sqrt{2}\,y+\sqrt{3}} dz\,dx\,dy \\
& + \int_{0}^{\sqrt{3}/2\,r} \int_{1/\sqrt{3}\,x}^{\sqrt{r^2-x^2}} \int_{\sqrt{2}\,y}^{-\sqrt{2}/\sqrt{2}\,x-1/\sqrt{2}\,y+\sqrt{3}} dz\,dy\,dx \\
& + \int_{-\sqrt{3}/2\,r}^{0} \int_{-1/\sqrt{3}\,x}^{\sqrt{r^2-x^2}} \int_{\sqrt{2}\,y}^{\sqrt{3}/\sqrt{2}\,x-1/\sqrt{2}\,y+\sqrt{3}} dz\,dy\,dx \\
& + \int_{-r/2}^{r/2} \int_{-\sqrt{r^2-y^2}}^{-\sqrt{3}|y|} \int_{-\sqrt{3}/\sqrt{2}\,x-1/\sqrt{2}\,y}^{\sqrt{3}/\sqrt{2}\,x-1/\sqrt{2}\,y+\sqrt{3}} dz\,dx\,dy \\
& + \int_{-\sqrt{3}/2\,r}^{0} \int_{-\sqrt{r^2-x^2}}^{1/\sqrt{3}\,x} \int_{-\sqrt{3}/\sqrt{2}\,x-1/\sqrt{2}\,y}^{\sqrt{2}\,y+\sqrt{3}} dz\,dy\,dx \\
& + \int_{0}^{\sqrt{3}/2\,r} \int_{-\sqrt{r^2-x^2}}^{-1/\sqrt{3}\,x} \int_{\sqrt{3}/\sqrt{2}\,x-1/\sqrt{2}\,y}^{\sqrt{2}\,y+\sqrt{3}} dz\,dy\,dx
\end{aligned}$$

The proposer computed this sum using Mathematica, and obtained the following result.

$$V(r) = \sqrt{3}\,\pi r^2 - 2\sqrt{6}\,r^3$$

We observe that, as expected,

$$\lim_{r\to 0} \frac{V(r)}{\sqrt{3}\,\pi r^2} = 1,$$

so that, again as expected, $r \approx 0 \;\Rightarrow\; V(r) \approx \sqrt{3}\,\pi r^2$.