Problem for 2015 November

Proposed by Dan Jurca

Compute $\sum_{k=1}^{\infty} \frac{1}{k(k+1/2)}$.

Solution by the proposer

We show that the series converges to $4 - \ln 16 = 1.227411\ldots$.
First, for each positive integer n let $H_n = 1 + 1/2 + 1/3 + \cdots + 1/n$, and recall that

$$\lim_{n\to\infty} (H_n - \ln n)$$

exists and equals a certain number, the *Euler-Mascheroni constant*, $\gamma \approx 0.57721$. We will need the following Lemma. $\lim_{n\to\infty} (H_{2n} - H_n) = \ln 2$.

Proof.

If $0 < \varepsilon$, then there exist positive integers n_1 and n_2 such that

$$\begin{aligned}
n_1 \le n &\Rightarrow\ |(H_{2n} - \ln 2n) - \gamma| < \varepsilon/2 \qquad \text{and}\\
n_2 \le n &\Rightarrow\ |(H_n - \ln n) - \gamma| < \varepsilon/2. \qquad \text{Hence}\\
n_1 \le n &\Rightarrow -\varepsilon/2 < H_{2n} - \ln 2n - \gamma < \varepsilon/2 \qquad \text{and}\\
n_2 \le n &\Rightarrow -\varepsilon/2 < \gamma - H_n + \ln n < \varepsilon/2; \qquad \text{whence (by addition)}\\
\max\{n_1, n_2\} \le n &\Rightarrow\ -\varepsilon < H_{2n} - H_n - \ln 2n + \ln n < \varepsilon, \qquad \text{so}\\
\max\{n_1, n_2\} \le n &\Rightarrow\ |(H_{2n} - H_n) - \ln 2| < \varepsilon,
\end{aligned}$$

proving the lemma.

Now since

$$1 \le k \ \Rightarrow\ \frac{1}{k(k+1/2)} = \frac{2}{k} - \frac{4}{2k+1},$$

computing the first few partial sums of the series and a little algebra suggests the following

Proposition. $1 \le n \ \Rightarrow\ S_n = \sum_{k=1}^{n} \frac{1}{k(k+1/2)} = 4 - 4(H_{2n} - H_n) - \frac{4}{2n+1}.$

Proof.

Clearly we have $S_1 = 1/(1\cdot 3/2) = 2/3 = 2 - 4/3 = 4 - 4(3/2 - 1) - 4/3 = 4 - 4(H_{2\cdot 1} - H_1) - 4/(2\cdot 1 + 1)$, so the assertion holds for $n = 1$. Next, if $2 \le n$ and

$$\begin{aligned}
S_{n-1} &= 4 - 4(H_{2n-2} - H_{n-1}) - 4/(2n-1), \qquad \text{then}\\
S_n &= 4 - 4(H_{2n-2} - H_{n-1}) - 4/(2n-1) + 1/(n(n+1/2))\\
&= 4 - 4(H_{2n-2} - H_{n-1}) - 4/(2n-1) + 2/n - 4/(2n+1)\\
&= 4 - 4(H_{2n-2} - H_{n-1}) - 4/(2n-1) - 4/2n + 4/2n + 2/n - 4/(2n+1)\\
&= 4 - 4(H_{2n} - H_n) - 4/(2n+1),
\end{aligned}$$

and the proposition follows by induction on n.

Hence by the proposition and the lemma

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1/2)} = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k(k+1/2)} = \lim_{n\to\infty} S_n = \lim_{n\to\infty} [(4 - 4(H_{2n} - H_n) - 4/(2n+1)] = 4 - 4\ln 2 = 4 - \ln 16.$$

Solution by Benjamin Thomas

$$\begin{aligned}
\frac{1}{k(k+1/2)} &= 2\,[1/k - 1/(k+1/2)]\\
&= 4\,[1/(2k) - 1/(2k+1)], \quad \text{so}\\
\sum_{k=1}^{\infty} \frac{1}{k(k+1/2)} &= 4\,[(1/2 - 1/3) + (1/4 - 1/5) + (1/6 - 1/7) + \cdots]\\
&= 4 - 4\,[1 - 1/2 + 1/3 - 1/4 + 1/5 - 1/6 + - \cdots]\\
&= 4 - 4\ln 2.
\end{aligned}$$

Also solved by Jan van Delden (the Netherlands), Massoud Malek, John M. Sayer, and Winston Teitler
John M. Sayer found the following formula on the internet.

$$\sum_{n=1}^{\infty} \frac{1}{n(n+\alpha)} = \frac{1}{\alpha}\,[\psi(\alpha+1) + \gamma]$$

where ψ is the psi (or digamma) function.