Problem for 2015 December

Proposed by Dan Jurca

Prove that the natural logarithm function is transcendental; *i.e.*, there do not exist polynomial functions p and q such that

$$0 < x \;\Rightarrow\; \log_e(x) = \frac{p(x)}{q(x)}.$$

Solution by the proposer

We suppose otherwise, and derive a contradiction. So suppose p and q are polynomial functions, and $0 < x \;\Rightarrow\; \log_e(x) = \ln x = p(x)/q(x)$. Obviously q is not a constant function, as otherwise ln is a polynomial function, and some higher order derivative of ln must be identically zero. Since

$$\lim_{\infty} \ln = \infty,$$

it follows that $1 \leq \text{degree}(q) < \text{degree}(p)$. Then by the division theorem (for polynomials) there exist polynomial functions Q and R such that

i. $p = q \cdot Q + R$, and
ii. $\text{degree}(R) < \text{degree}(q)$.

Since $\text{degree}(q) < \text{degree}(p)$, it follows that $1 < \text{degree}(Q)$; therefore

$$\lim_{x\to\infty} \frac{Q(x)}{x} = \infty \qquad \text{and also} \qquad \lim_{x\to\infty} \frac{R(x)}{x \cdot q(x)} = 0.$$

But then

$$\begin{aligned} 0 &= \lim_{x\to\infty} \frac{\ln x}{x} \\ &= \lim_{x\to\infty} \frac{p(x)/q(x)}{x} \\ &= \lim_{x\to\infty} \frac{q(x)\cdot Q(x) + R(x)}{x \cdot q(x)} \\ &= \lim_{x\to\infty} \frac{Q(x)}{x} + \lim_{x\to\infty} \frac{R(x)}{x \cdot q(x)} \\ &= \lim_{x\to\infty} \frac{Q(x)}{x} + 0 \\ &\neq 0, \end{aligned}$$

an obvious contradiction.

Also solved by Jan van Delden (the Netherlands; two solutions) and Winston Teitler