Problem for 2016 March

Proposed by Dan Jurca

Recall

i. if $x \in \mathbf{R},\ x \neq 0,$ and $a \in \mathbf{R}$, then one (usually) defines the *relative error* in approximating x with a to equal

$$\frac{x-a}{x}; \quad \text{and}$$

ii. $\lim\limits_{x\to\infty} x^{1/x} = 1.$

Let $G = \text{googol} = 10^{100}$, and find an approximation a of $\sqrt[G]{G} - 1$ with (the absolute value of the) relative error less than 10^{-6}.

Solution by the proposer

If $f : (0,\infty) \to \mathbf{R}$ by $f(x) = x^{1/x} - 1 = e^{\ln x/x} - 1$, then (since $e^z = 1 + z + z^2/2! + z^3/3! + z^4/4! + \cdots$)

$$f(x) = \frac{\ln x}{x} + \frac{1}{2!}\left(\frac{\ln x}{x}\right)^2 + \frac{1}{3!}\left(\frac{\ln x}{x}\right)^3 + \frac{1}{4!}\left(\frac{\ln x}{x}\right)^4 + \cdots, \quad \text{so}$$

$$\begin{aligned} 1 < x \ \Rightarrow\ \frac{\ln x}{x} < f(x) &= \frac{\ln x}{x}\cdot\left[1 + \frac{1}{2}\left(\frac{\ln x}{x}\right) + \frac{1}{6}\left(\frac{\ln x}{x}\right)^2 + \frac{1}{24}\left(\frac{\ln x}{x}\right)^3 + \cdots\right] \\ &< \frac{\ln x}{x}\cdot\left[1 + \left(\frac{\ln x}{2x}\right) + \left(\frac{\ln x}{2x}\right)^2 + \left(\frac{\ln x}{2x}\right)^3 + \cdots\right] \\ &= \frac{\ln x}{x}\cdot\frac{1}{1-(\ln x/2x)}; \quad \textit{i.e.,} \\ 1 < x \ \Rightarrow\ \frac{\ln x}{x} < f(x) &< \frac{\ln x}{x}\cdot\frac{2x}{2x-\ln x} = \frac{\ln x}{x}\cdot\left(1 + \frac{\ln x}{2x-\ln x}\right). \end{aligned}$$

Therefore

$$1 < x \ \Rightarrow\ 0 < f(x) - \ln x/x < \frac{\ln x}{x}\cdot\frac{2x}{2x-\ln x} - \frac{\ln x}{x} = \frac{\ln x}{x}\cdot\left[\frac{2x}{2x-\ln x} - 1\right] = \frac{\ln x}{x}\cdot\frac{\ln x}{2x-\ln x}.$$

Since $1 < x \ \Rightarrow\ \ln x/x < f(x)$, (so $1/f(x) < 1/(\ln x/x)$) it follows that

$$1 < x \ \Rightarrow\ 0 < \frac{f(x)-\ln x/x}{f(x)} < \frac{f(x)-\ln x/x}{\ln x/x} < \frac{\ln x}{2x-\ln x}.$$

Thus $1 < x \ \Rightarrow\ f(x) \approx \ln x/x$, with (absolute value of the) relative error less than $\ln x/(2x - \ln x)$.
In particular $\sqrt[G]{G} - 1 = f(G) \approx \ln G/G = \ln 10^{100}/10^{100} = 10^2 \ln 10/10^{100} = \ln 10 \times 10^{-98}$ with a relative error less than $\ln G/(2G - \ln G) < 10^{-97}$. Hence $\sqrt[G]{G} - 1 \approx a = \ln G/G = \ln 10 \times 10^{-98} \approx 2.302585 \times 10^{-98}$.

Also solved by Jan van Delden (the Netherlands)