Problem for 2016 April

Proposed by Dan Jurca

Find $\displaystyle\lim_{x\to\infty}\int_x^{x+1}|\sin(t^2)|\,dt$.

Solution by the proposer

The limit equals $2/\pi$. We shall need the following

Lemma. $1 \le n \;\Rightarrow\; \dfrac{1}{\sqrt{(n+1)\pi}} < \displaystyle\int_{\sqrt{n\pi}}^{\sqrt{(n+1)\pi}} |\sin(t^2)|\,dt < \dfrac{1}{\sqrt{n\pi}}$.

Proof.

For $t \in [\sqrt{n\pi}, \sqrt{(n+1)\pi}]$ let $u = t^2 - n\pi$; then $du = 2t\,dt$, so $dt = du/(2t) = du/(2\sqrt{u+n\pi})$. Therefore

$$\int_{\sqrt{n\pi}}^{\sqrt{(n+1)\pi}} \sin(t^2)\,dt = \frac{1}{2}\int_0^\pi \frac{\sin(u+n\pi)}{\sqrt{u+n\pi}}\,du = \frac{(-1)^n}{2}\int_0^\pi \frac{\sin u}{\sqrt{u+n\pi}}\,du.$$

Hence

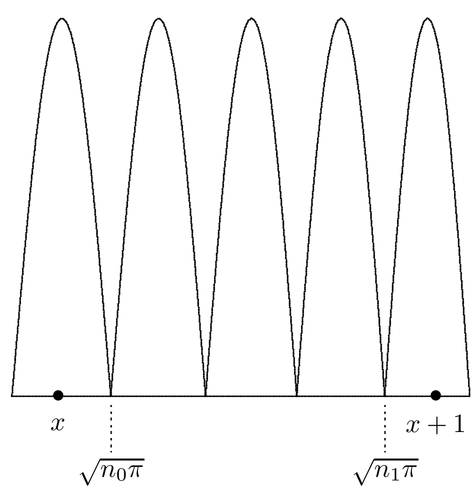
$$\int_{\sqrt{n\pi}}^{\sqrt{(n+1)\pi}} |\sin(t^2)|\,dt = \frac{1}{2}\int_0^\pi \frac{\sin u}{\sqrt{u+n\pi}}\,du.$$

Then since $u \in [0,\pi] \;\Rightarrow\; \sqrt{n\pi} \le \sqrt{u+n\pi} \le \sqrt{(n+1)\pi}$, with equality only at the endpoints, we have

$$\frac{1}{2\sqrt{(n+1)\pi}}\int_0^\pi \sin u\,du < \int_{\sqrt{n\pi}}^{\sqrt{(n+1)\pi}} |\sin(t^2)|\,dt < \frac{1}{2\sqrt{n\pi}}\int_0^\pi \sin u\,du,$$

and the lemma follows since $\displaystyle\int_0^\pi \sin u\,du = 2$.

The following sketch of (part of) the graph of $|\sin(t^2)|$ shows how we use the lemma. (Here $x = 6$.)



Now consider large x, say $3 \le x$. Let $n_0 = \lceil x^2/\pi \rceil$ and $n_1 = \lfloor (x+1)^2/\pi \rfloor$. Then $n_0 - 1 < x^2/\pi \le n_0$, so that $\sqrt{(n_0-1)\pi} < x \le \sqrt{n_0\pi}$; and $n_1 \le (x+1)^2/\pi < n_1 + 1$, so that $\sqrt{n_1\pi} \le x+1 < \sqrt{(n_1+1)\pi}$. Therefore there exist the following inclusion relations of intervals. (One checks that $\pi - 1/2 < x \;\Rightarrow\; n_0 < n_1$.)

$$[\sqrt{n_0\pi}, \sqrt{n_1\pi}] \;\subset\; [x, x+1] \;\subset\; [\sqrt{(n_0-1)\pi}, \sqrt{(n_1+1)\pi}]$$

Hence with $f(t) = |\sin(t^2)|$ (and since $0 \le f$)

$$\int_{\sqrt{n_0\pi}}^{\sqrt{n_1\pi}} f \le \int_x^{x+1} f < \int_{\sqrt{(n_0-1)\pi}}^{\sqrt{(n_1+1)\pi}} f$$

Next,

$$\int_{\sqrt{n_0\pi}}^{\sqrt{n_1\pi}} f = \sum_{i=1}^{n_1-n_0} \int_{\sqrt{(n_0-1+i)\pi}}^{\sqrt{(n_0+i)\pi}} f \qquad \text{and} \qquad \int_{\sqrt{(n_0-1)\pi}}^{\sqrt{(n_1+1)\pi}} f = \sum_{i=1}^{n_1-n_0+2} \int_{\sqrt{(n_0-2+i)\pi}}^{\sqrt{(n_0-1+i)\pi}} f.$$

By the lemma

$$1 \le i \le n_1 - n_0 \;\Rightarrow\; \frac{1}{\sqrt{n_1\pi}} \le \frac{1}{\sqrt{(n_0+i)\pi}} < \int_{\sqrt{(n_0-1+i)\pi}}^{\sqrt{(n_0+i)\pi}} f \quad \text{and}$$

$$1 \le i \le n_1 - n_0 + 2 \;\Rightarrow\; \int_{\sqrt{(n_0-2+i)\pi}}^{\sqrt{(n_0-1+i)\pi}} f < \frac{1}{\sqrt{(n_0-2+i)\pi}} \le \frac{1}{\sqrt{(n_0-1)\pi}},$$

so from the sums above

$$\frac{n_1-n_0}{\sqrt{n_1\pi}} < \int_x^{x+1} f < \frac{n_1-n_0+2}{\sqrt{(n_0-1)\pi}}.$$

Since

$$\frac{(x+1)^2}{\pi} - 1 < n_1 \text{ and } n_0 < \frac{x^2}{\pi} + 1, \text{ we have } \frac{(x+1)^2}{\pi} - 1 - \frac{x^2}{\pi} - 1 = \frac{2x - 2\pi + 1}{\pi} < n_1 - n_0,$$

and since

$$\frac{1}{x+1} \le \frac{1}{\sqrt{n_1\pi}} \text{ it follows that } \frac{2x-2\pi+1}{\pi(x+1)} < \frac{n_1-n_0}{\sqrt{n_1\pi}}.$$

Similarly, since

$$n_1 \le \frac{(x+1)^2}{\pi} \text{ and } \frac{x^2}{\pi} \le n_0, \text{ we have } n_1 - n_0 + 2 \le \frac{(x+1)^2}{\pi} - \frac{x^2}{\pi} + 2 = \frac{2x+2\pi+1}{\pi},$$

and since

$$\frac{x^2}{\pi} - 1 \le n_0 - 1 \;\left(\text{so } \frac{1}{\sqrt{(n_0-1)\pi}} \le \frac{1}{\sqrt{x^2-\pi}}\right), \text{ we have } \frac{n_1-n_0+2}{\sqrt{(n_0-1)\pi}} \le \frac{2x+2\pi+1}{\pi\sqrt{x^2-\pi}}.$$

Thus

$$3 \le x \;\Rightarrow\; \frac{2x-2\pi+1}{\pi(x+1)} < \int_x^{x+1} |\sin(t^2)|\,dt \le \frac{2x+2\pi+1}{\pi\sqrt{x^2-\pi}},$$

and the limit in question follows from the "squeeze theorem".

Remark. In fact, one can show (from the inequalities above) that

$$3 \le x \;\Rightarrow\; \frac{2}{\pi}\left(1 - \frac{4}{x+1}\right) < \int_x^{x+1} |\sin(t^2)|\,dt < \frac{2}{\pi}\left(1 + \frac{4}{x-1}\right).$$

Also solved by Jan van Delden (The Netherlands) and Winston Teitler (numerically)