Problem for 2016 June

Communicated by Dan Jurca

The following is well known; the proposer learned of it only recently.

Prove that if A is a square real or complex matrix, then $\det(e^A) = e^{\mathrm{trace}(A)}$.

Solution by Dan Jurca

Suppose A is $n \times n$. If $\lambda \in \mathrm{spec}(A)$, (*i.e.*, λ is an eigenvalue of A), then for some nonzero $\mathbf{x} \in \mathbf{R}^n$ or $\mathbf{x} \in \mathbf{C}^n$, $A\mathbf{x} = \lambda\mathbf{x}$. Hence

$$\begin{aligned} e^A\mathbf{x} &= \left(I_n + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots\right)\mathbf{x} \\ &= \mathbf{x} + A\mathbf{x} + \frac{A^2\mathbf{x}}{2!} + \frac{A^3\mathbf{x}}{3!} + \cdots \\ &= 1 \cdot \mathbf{x} + \lambda \cdot \mathbf{x} + \frac{\lambda^2}{2!} \cdot \mathbf{x} + \frac{\lambda^3}{3!} \cdot \mathbf{x} + \cdots \\ &= \left(1 + \lambda + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \cdots\right)\mathbf{x} \\ &= e^\lambda\mathbf{x}. \end{aligned}$$

So if the characteristic polynomial $p(x) = \det(xI_n - A)$ splits over $\mathbf{C}$ as $(x-\lambda_1)(x-\lambda_2)\cdots(x-\lambda_n)$, then for each of the (not necessarily distinct) eigenvalues λ_j, $j = 1, \ldots, n$ of A, e^{λ_j} is an eigenvalue of e^A. It follows that $\mathrm{spec}(e^A) = \{e^\lambda \mid \lambda \in \mathrm{spec}(A)\}$. (This is non-trivial: see Theorem 6 on page 312 of *The Theory of Matrices* by Peter Lancaster and Miron Tismenetsky.) Since the determinant of a matrix equals the product of the eigenvalues of the matrix,

$$\begin{aligned} \det(e^A) &= e^{\lambda_1}e^{\lambda_2}e^{\lambda_3}\cdots e^{\lambda_n} \\ &= e^{\lambda_1+\lambda_2+\lambda_3+\cdots+\lambda_n} \\ &= e^{\mathrm{trace}(A)}, \end{aligned}$$

since the trace of a matrix equals the sum of the eigenvalues of the matrix.

Also solved by Massoud Malek and Benjamin Thomas