Problem for 2016 July

Communicated by Dan Jurca

The following problem appears as problem 2, section 11, on page 96 of *Topology* by James Dugundji.

Let $p(x)$ be a polynomial on E^1. Show that $x \to p(x)$ is a closed map of E^1.

Solution by Dan Jurca

Here E^1 is the set $\mathbf{R}$ with the usual (euclidean) topology. If p is a constant function (*i.e.*, the degree of $p(x)$ is less than 1) then p is a closed map since points in $\mathbf{R}$ are closed. If $p(x) = c_1 x + c_0$, where $c_1 \neq 0$, then p is a homeomorphism with inverse $x \mapsto (x - c_0)/c_1$, so is closed. Hence we suppose the degree of $p(x)$ is at least 2.

Lemma. If $r \in \mathbf{R}$, $f : (-\infty, r] \to \mathbf{R}$ is continuous and strictly increases, then f is a closed map.

Proof.

If $a < b \leq r$, then (as is easily shown) $f((a, b)) = (f(a), f(b))$. Then since each open set in $\mathbf{R}$ is a union of open intervals, it follows that f is an open map. Since f is a bijection onto its image, it then follows that f is also closed, and the lemma is proved.

Remark. In the same way one shows a similar result if f decreases; and similarly if $f : [r, \infty) \to \mathbf{R}$.

Now suppose $p(x)$ is a polynomial and the degree d of $p(x)$ is at least 2. Let $Z = \{r \in \mathbf{R} \mid p'(r) = 0\}$, the set of zeros of the derivative p' of p. Clearly if $Z = \emptyset$ (in which case d is necessarily odd), then p either strictly increases or strictly decreases, so is a homeomorphism, and is therefore closed. Next suppose $Z = \{r_1, r_2, \ldots, r_k\}$, where $r_1 < r_2 < \cdots < r_k$, and $k < d$. If $k = 1$, then by the lemma (and the remark which follows it)

$$F \subset \mathbf{R} \;\Rightarrow\; p(F) = p|_{(-\infty, r_1]}(F \cap (-\infty, r_1]) \cup p|_{[r_1, \infty)}(F \cap [r_1, \infty)),$$

so if F is a closed subset of $\mathbf{R}$, then $p(F)$ is closed. If $2 \leq k$, then since for $i = 1, \ldots, k-1$ the interval $[r_i, r_{i+1}]$ is compact, it follows that the restriction $p|_{[r_i, r_{i+1}]}$ is closed. Then since $F \subset \mathbf{R} \;\Rightarrow$

$$p(F) = p|_{(-\infty, r_1]}(F \cap (-\infty, r_1]) \cup p|_{[r_1, r_2]}(F \cap [r_1, r_2]) \cup \cdots \cup p|_{[r_{k-1}, r_k]}(F \cap [r_{k-1}, r_k]) \cup p|_{[r_k, \infty)}(F \cap [r_k . \infty)),$$

it follows that if F is closed, then $p(F)$, a finite union of closed sets, is also closed; so p is a closed map.

Remark. A polynomial function is not necessarily open: if $p(x) = x^2$, then $p((-1, 1)) = [0, 1)$, which is not an open set.