Problem for 2016 August

Proposed by Benjamin Thomas

Find the general solution in terms of elementary functions of the following differential equation.

$$2x(x+1)\frac{d^2y}{dx^2} + 3(x+1)\frac{dy}{dx} - y = 0$$

Solution by Dan Jurca

Observing that $x = 0$ is a regular singular point one tries, using a method suggested by a theorem of Georg Frobenius, a series solution of the following form, where r is to be determined.

$$y = c_0x^r + c_1x^{1+r} + c_2x^{2+r} + c_3x^{3+r} + \cdots$$

Substituting this into the DE one finds the coefficient of x^{r-1} to equal $(2r^2 + r)c_0$, and equating this to 0 we find $r = -1/2$ or $r = 0$. Hence we try

$$\begin{aligned} y_0 &= x^{-1/2}\left(a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots\right) \quad \text{and} \\ y_1 &= b_0 + b_1x + b_2x^2 + b_3x^3 + \cdots. \end{aligned}$$

Substituting first the series for y_0 into the DE we find that $a_1 = a_0$ and $a_2 = a_3 = \cdots = 0$; therefore $y_0 = a_0\left(x^{-1/2} + x^{1/2}\right)$.

Substituting the series for y_1 into the DE we find

$$y_1 = b_0\left(1 + \frac{1}{1\cdot 3}x - \frac{1}{3\cdot 5}x^2 + \frac{1}{5\cdot 7}x^3 + - \cdots\right),$$

and writing

$$\frac{1}{(2n-1)(2n+1)} = \frac{1}{2}\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right),$$

we recognize the series for $\arctan(\sqrt{x})$, and find that

$$y_1 = \frac{b_0}{2}\left(1 + (x+1)\frac{\arctan(\sqrt{x})}{\sqrt{x}}\right).$$

$\{y_0, y_1\}$ seems linearly independent in the interval $(0,\infty)$, but here is a proof. If $\{y_0, y_1\}$ is linearly dependent, then there exists a constant k such that $y_1 = ky_0$; therefore

$$\begin{aligned} 1 + \frac{(x+1)\arctan(\sqrt{x})}{\sqrt{x}} &= k\cdot\frac{x+1}{\sqrt{x}}, \quad \text{so} \\ \arctan(\sqrt{x}) &= \frac{\sqrt{x}}{x+1}\left(k\cdot\frac{x+1}{\sqrt{x}} - 1\right) = k - \frac{\sqrt{x}}{x+1}. \end{aligned}$$

Since $\arctan(\sqrt{x}) \to \pi/2$ as $x \to \infty$, it follows that $k = \pi/2$; however, (with $x = 1$) $\pi/4 \neq \pi/2 - 1/2$, so no such k exists, and $\{y_0, y_1\}$ is linearly independent.

Therefore the general solution of the differential equation equals the following.

$$0 < x \ \Rightarrow\ y(x) = c_0\frac{x+1}{\sqrt{x}} \ + \ c_1\left(1 + \frac{x+1}{\sqrt{x}}\cdot\arctan(\sqrt{x})\right)$$

This can be (and has been) checked with the following Mathematica program.

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y[x_] = c0(x+1)/Sqrt[x]+c1(1+(x+1)/Sqrt[x]ArcTan[Sqrt[x]])
d1[x_] = D[y[x],x]
d2[x_] = D[d1[x],x]
Simplify[2x(x+1)d2[x]+3(x+1)d1[x]-y[x]]
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Also solved by Massoud Malek and the proposer