Problem for 2016 August

Proposed by Benjamin Thomas

Find the general solution in terms of elementary functions:

$$2x\left(1+x\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+3\left(1+x\right)\frac{\mathrm{d}y}{\mathrm{d}x}-y=0$$

Solution by the proposer

Any analytic solution can be expressed as a power series of the form $\sum_{n=0}^{\infty} a_n x^n$.

Let

$$y=\sum_{n=0}^{\infty} a_n x^n$$
$$\frac{\mathrm{d}y}{\mathrm{d}x}=\sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\sum_{n=2}^{\infty} n\left(n-1\right) a_n x^{n-2}$$

We notice the immediate consequence: $\sum_{n=0}^{\infty} n a_n x^n = x\frac{\mathrm{d}y}{\mathrm{d}x}$

Plugging these back into our differential equation and rearranging terms we arrive at $\sum_{n=0}^{\infty}\left[\left(2n-1\right)\left(n+1\right)a_n+\left(2n+3\right)\left(n+1\right)a_{n+1}\right]x^n=0$.

In order to find the sequence, a_n, we can set each coefficient of x^n equal to 0.

$$\left(2n-1\right)\left(n+1\right)a_n+\left(2n+3\right)\left(n+1\right)a_{n+1}=0$$
$$\left(2n-1\right)a_n+\left(2n+3\right)a_{n+1}=0$$

This allows us to arrive at

$$\sum_{n=0}^{\infty}\left(2n-1\right)a_n x^n+\sum_{n=0}^{\infty}\left(2n+3\right)a_{n+1}x^n=0$$

and finally

$$2x\left(1+x\right)\frac{\mathrm{d}y}{\mathrm{d}x}+\left(1-x\right)y=a_0$$

for some arbitrary constant, $a_0=y\left(0\right)$.

Putting this in standard form and using the integrating factor $\mu=\frac{\sqrt{x}}{1+x}$ we arrive at our solution.

$$y=\frac{a_0}{2}\left(1+\frac{1+x}{\sqrt{x}}\tan^{-1}\left(\sqrt{x}\right)\right)+C\frac{1+x}{\sqrt{x}}$$
$$y=C_1\left(1+\frac{1+x}{\sqrt{x}}\tan^{-1}\left(\sqrt{x}\right)\right)+C_2\frac{1+x}{\sqrt{x}}$$