Problem for 2016 September

Communicated by Dan Jurca

This problem appeared on the internet. Prove the following proposition.

If

- a is a positive integer, and
- b is a positive integer, and
- $ab+1$ divides a^2+b^2,

then $\dfrac{a^2+b^2}{ab+1}$ is a perfect square.

Solution by Dan Jurca

Suppose a is an integer, b is an integer, and $ab+1$ divides a^2+b^2. If $q=(a^2+b^2)/(ab+1)$, then obviously q equals a square if either a or b equals 0; and if $a=b$, then

$$q=\frac{a^2+b^2}{ab+1}=\frac{2a^2}{a^2+1}=2-\frac{2}{a^2+1}<2,$$

and since q equals a nonnegative integer, $q=0$ or $q=1$, so q is a square. Thus we may and do assume from now on that $0<a<b$, $ab+1$ divides a^2+b^2, and $q=(a^2+b^2)/(ab+1)$.

Let $a'=qa-b$ and $b'=a$. Then a' is an integer, b' is an integer, and

$$0<a<b \;\Rightarrow\; a^3<a^2b \;\Rightarrow\; a^3+ab^2<a^2b+ab^2+a+b \;\Rightarrow\; a(a^2+b^2)<(ab+1)(a+b) \;\Rightarrow\; qa<a+b \;\Rightarrow\; a'<b'.$$

Thus $a'<b'=a$, so $a'<a$; and $b'=a<b$; *i.e.*, $a'<a$ and $b'<b$. Moreover

$$\begin{aligned}\frac{(a')^2+(b')^2}{a'b'+1} &= \frac{q^2a^2-2qab+b^2+a^2}{qa^2-ab+1}=\frac{q^2a^2-qab+q+a^2+b^2-qab-q}{qa^2-ab+1}\\ &= \frac{q(qa^2-ab+1)+[(a^2+b^2)-q(ab+1)]}{qa^2-ab+1}=q\frac{qa^2-ab+1}{qa^2-ab+1}+0\\ &= q.\end{aligned}$$

(If $a'b'=-1$, then $a'=-1$ and $b'=1=a$; but then $a'=qa-b=q-b=-1$, so $q=(1^2+b^2)/(b+1)=b-1$, but $b^2+1\neq b^2-1$, so this does not occur; *i.e.*, $a'b'+1\neq 0$.)

If one iterates this (with a' in place of a and b' in place of b) one obtains decreasing sequences of integral a's and b's, but always $(a^2+b^2)/(ab+1)=q$. Now suppose that $0<a<b$, but $a'\leq 0$; we will show that in fact then $a'=0$. (That is, a is the last strictly positive term in the sequence of iterates, and the next term, a', is less than or equal to 0.) Then, as observed above, q equals a square.

So suppose $0<a<b$, $(a^2+b^2)/(ab+1)=q$ is a positive integer, but $a'=qa-b\leq 0$. If $a=1$, then $q=(1+b^2)/(b+1)$; but b is a positive integer and

$$q=\frac{b^2+1}{b+1}=b-1+\frac{2}{b+1}$$

and this is an integer if and only if $b=1$, and since $a<b$, we have $b\neq 1$. Thus $1<a$. But then since $b'=a$, it follows that $1<b'$. Hence if $a'\leq -1$, then $a'b'<-1$, so that $a'b'+1<0$, and $q=((a')^2+(b')^2)/(a'b'+1)<0$, a contradiction. Therefore $-1<a'$, and since $a'\leq 0$, it follows that $a'=0$, and $q=(b')^2=a^2$ is a square.

Remark. One can produce all such a and b as follows.

Let n be an integer, and consider sequences $(a_k)_{k=0}^{\infty}$ and $(b_k)_{k=0}^{\infty}$ as follows.

$$a_0=0,\quad b_0=n;\qquad 1\leq k \;\Rightarrow\; a_k=b_{k-1},\quad b_k=n^2b_{k-1}-a_{k-1}$$

Then $0\leq k \;\Rightarrow\; \dfrac{a_k^2+b_k^2}{a_kb_k+1}=n^2$. Moreover, if a and b are positive integers and $ab+1$ divides a^2+b^2, then there exist n and k such that $a=a_k$ and $b=b_k$ where (a_k) and (b_k) are the sequences defined above.

Also solved by Benjamin Thomas