Problem for 2016 November

Proposed by Dan Jurca

Let A equal the infinite matrix $A = (a_{ij})_{0 \le i,\ 0 \le j}$ where

$$0 \le i,\ 0 \le j \ \Rightarrow\ a_{ij} = \frac{i^2 + j^2 + 2ij + 3i + j}{2}.$$

Prove that each natural number (nonnegative integer) appears exactly once as an entry in A.

Solution by the proposer

The first few rows and columns of A appear as follows.

$$\begin{array}{rrrrrrc} 0 & 1 & 3 & 6 & 10 & 15 & \dots \\ 2 & 4 & 7 & 11 & 16 & 22 & \dots \\ 5 & 8 & 12 & 17 & 23 & 30 & \dots \\ 9 & 13 & 18 & 24 & 31 & 39 & \dots \\ 14 & 19 & 25 & 32 & 40 & 49 & \dots \\ 20 & 26 & 33 & 41 & 50 & 60 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

For each $k,\ 0 \le k$, let t_k equal the k-th triangular number, $t_k = k(k+1)/2$, and let the k-th diagonal consist of the $k+1$ entries $a_{0,k},\ a_{1,k-1},\ a_{2,k-2},\ \dots,\ a_{k,0}$; *i.e.*, the a_{ij} where $i + j = k$. Then if $i + j = k$,

$$a_{ij} = \frac{i^2 + j^2 + 2ij + 3i + j}{2} = \frac{i^2 + 2ij + j^2 + i + j + 2i}{2} = \frac{(i+j)^2 + (i+j)}{2} + i = \frac{k^2 + k}{2} + i = \frac{k(k+1)}{2} + i,$$

so that the entries in the k-th diagonal increase (by 1 as i increases from 0 to k by 1) from

$$\begin{aligned} a_{0k} &= \frac{(0+k)(0+k+1)}{2} + 0 = \frac{k(k+1)}{2} \\ &= t_k \qquad \text{to} \\ a_{k0} &= \frac{(k+0)(k+0+1)}{2} + k = \frac{k^2 + 3k}{2} = \frac{k^2 + 3k + 2 - 2}{2} = \frac{(k+1)(k+2) - 2}{2} = \frac{(k+1)(k+2)}{2} - 1 \\ &= t_{k+1} - 1, \end{aligned}$$

and since the sequence $(t_k)_{k=0}^{\infty}$ strictly increases, no entry in A appears twice.

Next, if $0 \le n$ and $k = \lfloor(\sqrt{8n+1} - 1)/2\rfloor$, then $0 \le k$ and, first,

$$\begin{aligned} \frac{\sqrt{8n+1} - 1}{2} - 1 &< k \qquad \text{so} \\ \sqrt{8n+1} - 1 &< 2k + 2 \\ \sqrt{8n+1} &< 2k + 3 \\ 8n + 1 &< 4k^2 + 12k + 9 \\ 8n &< 4k^2 + 12k + 8 \\ n &< \frac{k^2 + 3k + 2}{2} = \frac{(k+1)(k+2)}{2} = t_{k+1}, \quad \text{and, second,} \\ k &\le \frac{\sqrt{8n+1} - 1}{2} \qquad \text{so} \\ 2k + 1 &\le \sqrt{8n+1} \\ 4k^2 + 4k &\le 8n \\ t_k = \frac{k(k+1)}{2} &\le n; \end{aligned}$$

i.e., $t_k \le n \le t_{k+1} - 1$, so that n appears in the k-th diagonal. In fact, if $i = n - k(k+1)/2$ and $j = k - i$, then $a_{ij} = n$.

Also solved by John Sayer, Winston Teitler, and Benjamin Thomas