Problem for 2016 December

Communicated by Dan Jurca

If U is a set and $\Delta : \mathcal{P}(U)\times\mathcal{P}(U) \to \mathcal{P}(U)$ by $(A,B) \in \mathcal{P}(U)\times\mathcal{P}(U) \;\Rightarrow\; \Delta(A,B) = A\Delta B = (A\cup B)-(A\cap B)$, then Δ is an associative binary operation. The problem is to find a "nice" proof; *i.e.*, a nice proof that $A\Delta(B\Delta C) = (A\Delta B)\Delta C$.

Solution by Dan Jurca

For each subset $A \subset U$ there exists the function (the characteristic function of A) $\chi_A\colon U \to \{0,1\}$ by

$$x \in U \;\Rightarrow \chi_A(x) = \begin{cases} 0 & \text{if } x \notin A \\ 1 & \text{if } x \in A \end{cases}$$

and the function $\chi : \mathcal{P}(U) \to 2^U = \{0,1\}^U$ by $A \subset U \;\Rightarrow \chi(A) = \chi_A$ is a bijection. The following are immediate, where $A \subset U \;\Rightarrow\; A^c = U - A$, the complement of A.

i. $\chi_{A\cap B} = \chi_A\chi_B$
ii. $\chi_{A\cup B} = \chi_A + \chi_B - \chi_A\chi_B$
iii. $\chi_{A^c} = 1 - \chi_A$.

Lemma 1. $\chi_{A-B} = \chi_A - \chi_A\chi_B$.

Proof.

$$\chi_{A-B} = \chi_{A\cap B^c} = \chi_A\chi_{B^c} = \chi_A\,(1-\chi_B) = \chi_A - \chi_A\chi_B.$$

Lemma 2. $\chi_{A\Delta B} = \chi_A + \chi_B - 2\,\chi_A\chi_B$.

Proof.

Since $A\Delta B = (A\cup B) - (A\cap B)$, $\chi_A\chi_A = \chi_{A\cap A} = \chi_A$, and $\chi_A\chi_B = \chi_{A\cap B} = \chi_{B\cap A} = \chi_B\chi_A$, by lemma 1

$$\begin{aligned}
\chi_{A\Delta B} &= \chi_{(A\cup B)-(A\cap B)} \\
&= \chi_{A\cup B} - \chi_{A\cup B}\chi_{A\cap B} \\
&= \chi_A + \chi_B - \chi_A\chi_B - (\chi_A + \chi_B - \chi_A\chi_B)\,\chi_A\chi_B \\
&= \chi_A + \chi_B - \chi_A\chi_B - \chi_A\chi_B - \chi_A\chi_B + \chi_A\chi_B \\
&= \chi_A + \chi_B - 2\,\chi_A\chi_B,
\end{aligned}$$

as claimed.

Therefore by lemma 2

$$\begin{aligned}
\chi\,(A\Delta(B\Delta C)) &= \chi_{A\Delta(B\Delta C)} \\
&= \chi_A + \chi_{B\Delta C} - 2\,\chi_A\chi_{B\Delta C} \\
&= \chi_A + \chi_B + \chi_C - 2\,\chi_B\chi_C - 2\,\chi_A\,(\chi_B + \chi_C - 2\,\chi_B\chi_C) \\
&= \chi_A + \chi_B + \chi_C - 2\,\chi_B\chi_C - 2\,\chi_A\chi_B - 2\,\chi_A\chi_C + 4\,\chi_A\chi_B\chi_C \\
&= \chi_A + \chi_B - 2\,\chi_A\chi_B + \chi_C - 2\,\chi_A\chi_C - 2\,\chi_B\chi_C + 4\,\chi_A\chi_B\chi_C \\
&= \chi_A + \chi_B - 2\,\chi_A\chi_B + \chi_C - 2(\chi_A + \chi_B - 2\,\chi_A\chi_B)\,\chi_C \\
&= \chi_{A\Delta B} + \chi_C - 2\,\chi_{A\Delta B}\chi_C \\
&= \chi_{(A\Delta B)\Delta C} \\
&= \chi\,((A\Delta B)\Delta C)\,,
\end{aligned}$$

and since χ is injective, it follows that $A\Delta(B\Delta C) = (A\Delta B)\Delta C$.

Also solved by John Sayer and Winston Teitler