Problem for 2017 January

Proposed by Dan Jurca

Determine, with proof, $\lim_{n\to\infty} \int_0^1 \sqrt[n]{1-x^n}\,dx$.

Solution 1 by the proposer

The limit equals, of course, 1. We shall use the following well-known

Lemma. If $u < v$, $f : [u, v] \to \mathbf{R}$ is continuous, and $u < x < v \Rightarrow f''(x) < 0$, then if $u \le a < b \le v$, no point on the curve defined by $y = f(x)$ is below the chord from $(a, f(a))$ to $(b, f(b))$.

Proof.

If $h : [a, b] \to \mathbf{R}$ by

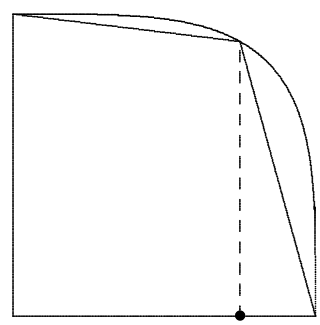
$$h(x) = f(x) - \left[\frac{f(b) - f(a)}{b - a}(x - a) + f(a)\right],$$

the assertion of the lemma is that $a \le x \le b \Rightarrow 0 \le h(x)$, which we prove as follows. Clearly $h(a) = 0 = h(b)$; we shall assume there exists t such that $a < t < b$, $h(t) < 0$, and derive a contradiction. By the Mean Value Theorem there exists ξ_1 such that $a < \xi_1 < t$ and $h(a) - h(t) = (a - t)h'(\xi_1)$; and there exists ξ_2 such that $t < \xi_2 < b$ and $h(t) - h(b) = (t - b)h'(\xi_2)$. Therefore $h'(\xi_1) = -h(t)/(a - t) < 0 < h(t)/(t - b) = h'(\xi_2)$, so that $\xi_1 < \xi_2$ and $h'(\xi_1) < h'(\xi_2)$. However, since $h'' = f'' < 0$, it follows that h' decreases, and this contradiction shows that no such t exists; therefore $0 \le h$.

Now suppose $1 \le n$ and let $f_n : [0, 1] \to \mathbf{R}$ by $f_n(x) = \sqrt[n]{1 - x^n}$. Then $0 < x < 1 \Rightarrow f_n''(x) < 0$ so by the lemma we find

$$\int_0^{1-1/n}\left[\frac{f_n(1-1/n) - f_n(0)}{(1-1/n) - 0}(x - 0) + f_n(0)\right] dx \, +$$
$$\int_{1-1/n}^1\left[\frac{f_n(1) - f_n(1-1/n)}{1 - (1-1/n)}\,(x - (1 - 1/n)) + f_n(1-1/n)\right] dx < \int_0^{1-1/n} f_n + \int_{1-1/n}^1 f_n = \int_0^1 f_n.$$

This says simply that the integral from 0 to 1 of f_n is greater than the sum of the areas of the trapezoid and the triangle in the following sketch (for $n = 4$)



and that sum of areas

$$\begin{aligned}
&= 1/2(1-1/n)\,[1 + f_n(1-1/n)] \;+\; 1/2(1/n)f_n(1-1/n)\\
&= 1/2\,[1 + f_n(1-1/n) - 1/n]\\
&= 1/2\left[1 + \sqrt[n]{1-(1-1/n)^n} - 1/n\right].
\end{aligned}$$

Therefore

$$1 \le n \;\Rightarrow\; 1/2\left[1 + \sqrt[n]{1-(1-1/n)^n} - 1/n\right] < \int_0^1 f_n < 1$$

It is well-known and easy to show that the sequence $((1 - 1/n)^n)_n^\infty$ increases and converges to $1/e$. Therefore

$$1 \le n \Rightarrow \sqrt[n]{1-1/e} < \sqrt[n]{1-(1-1/n)^n}, \quad \text{so}$$
$$1 \le n \Rightarrow 1/2\left[1 + \sqrt[n]{1-1/e} - 1/n\right] < \int_0^1 f_n < 1,$$

and since $0 < 1 - 1/e < 1$, it follows that $\lim_{n\to\infty} \sqrt[n]{1-1/e} = 1$, so by the "squeeze theorem" $\lim_{n\to\infty} \int_0^1 f_n = 1$.

Solution 2 by the proposer

Using the substitution $t = 1 - x^n$ we find

$$\begin{aligned}
\int_0^1 \sqrt[n]{1-x^n}\,dx &= \frac{1}{n}\int_0^1 t^{1/n}(1-t)^{1/n-1}\,dt \\
&= \frac{1}{n}\int_0^1 t^{(1/n+1)-1}(1-t)^{(1/n)-1}\,dt \\
&= 1/n \cdot B(1/n+1, 1/n) \qquad \text{where } B \text{ is the Beta function} \\
&= 1/n \cdot \frac{\Gamma(1/n+1)\Gamma(1/n)}{\Gamma(1/n+1+1/n)} \\
&= 1/n \cdot \Gamma(1/n) \cdot \frac{\Gamma(1+1/n)}{\Gamma(1+2/n)} \\
&\to 1 \text{ as } n \to \infty \text{ since } \lim_{x\to 0^+} x\Gamma(x) = \lim_{x\to 0^+} \Gamma(x+1) = \Gamma(1) = 1,
\end{aligned}$$

and the limit equals 1, as before.

Solution by Charles Burnette (graduate student), Drexel University

For each nonnegative integer n, $t \in \mathbf{R} \;\Rightarrow\; 1 - t^n = (1-t)(1+t+t^2+\cdots+t^{n-1})$; and since $x \in (0,1) \;\Rightarrow\; 0 < \sqrt[n]{1-x^n} < 1$, we find

$$\begin{aligned}
0 &< 1 - \int_0^1 \sqrt[n]{1-x^n}\,dx \\
&= \int_0^1 (1 - \sqrt[n]{1-x^n})\,dx \\
&= \int_0^1 \frac{x^n}{1 + (1-x^n)^{\frac{1}{n}} + (1-x^n)^{\frac{2}{n}} + \cdots + (1-x^n)^{\frac{n-1}{n}}}\,dx \\
&< \int_0^1 x^n\,dx \\
&= \frac{1}{n+1} \to 0 \text{ as } n \to \infty.
\end{aligned}$$

Hence $\lim_{n\to\infty} \int_0^1 \sqrt[n]{1-x^n}\,dx = 1.$

Lemma. If $\psi : (1,\infty) \to \mathbf{R}$ by $\psi(x) = \ln(1-1/x) + 1/(x-1)$, then $0 < \psi$.
Proof.

$$1 < x \;\Rightarrow\; \psi'(x) = \frac{1}{1-1/x}\cdot\frac{1}{x^2} - \frac{1}{(x-1)^2} = \frac{1}{x^2-x} - \frac{1}{(x-1)^2} = \frac{1}{x(x-1)} - \frac{1}{(x-1)^2} = \frac{x-1-x}{x(x-1)^2}$$
$$= -\frac{1}{x(x-1)^2} < 0,$$

so ψ strictly decreases. Since also $\lim_{\infty}\psi = 0$, it follows that $1 < x \;\Rightarrow\; 0 < \psi(x)$.

Lemma. If $\varphi : (1,\infty) \to \mathbf{R}$ by $\varphi(x) = (1-1/x)^x$, then φ strictly increases.
Proof.
Since $\ln\varphi(x) = x\ln(1-1/x)$,

$$1 < x \;\Rightarrow\; \frac{1}{\varphi(x)}\varphi'(x) = \ln\left(1-\frac{1}{x}\right) + \frac{x}{1-1/x}\cdot\frac{1}{x^2} = \ln\left(1-\frac{1}{x}\right) + \frac{1}{x-1},$$

which by the previous lemma is positive for each x; therefore (since $0 < \varphi$) $0 < \varphi'$, and φ strictly increases.

Lemma. With φ as in the previous lemma $\lim_{\infty}\varphi = 1/e$.
Proof.

$$\ln\varphi(x) = x\ln(1-1/x) = \frac{\ln(1-1/x)}{1/x}$$

so by l'Hospital's rule

$$\lim_{x\to\infty}\ln\varphi(x) = \lim_{x\to\infty}\frac{\dfrac{1}{1-1/x}\cdot\dfrac{1}{x^2}}{-\dfrac{1}{x^2}} = -\lim_{x\to\infty}\frac{x}{x-1} = -1,$$

and since $\lim_{\infty}\ln\varphi = -1$, it follows that $\lim_{\infty}\varphi = e^{-1} = 1/e$.