**Problem for 2017 March**

Proposed by Dan Jurca

Prove that if $S$ is a set, $\mathcal{P}(S)$ is the set of subsets of $S$, $F : \mathcal{P}(S) \to \mathcal{P}(S)$, and for each $A \subset S$ and $B \subset S$, $A \subset B \;\Rightarrow\; F(A) \subset F(B)$, then there exists a subset $X$ of $S$ such that $F(X) = X$.

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Solution by the proposer

Let $\mathcal{Q} = \{A \subset S \mid A \subset F(A)\}$. Then $\mathcal{Q} \subset \mathcal{P}(S)$, and $\mathcal{Q} \neq \emptyset$, since $\emptyset \in \mathcal{Q}$. Let $X = \cup\mathcal{Q}$; *i.e.*,

$$X = \cup\{A \subset S \mid A \subset F(A)\}.$$

We show $F(X) = X$.

First we show $X \subset F(X)$. For if $x \in X$, then there exists $A \subset S$ such that $x \in A$, $A \subset X$, and $A \subset F(A)$; therefore $x \in A \subset F(A) \subset F(X)$, so $x \in F(X)$; therefore $X \subset F(X)$.

Next, since $X \subset F(X)$, it follows that $F(X) \subset F(F(X))$, so that $F(X) \in \mathcal{Q}$. Therefore, since $A \in \mathcal{Q} \;\Rightarrow\; A \subset X$, $F(X) \subset X$.

Then since $X \subset F(X)$ and $F(X) \subset X$, it follows that $F(X) = X$.

Remark. This is a special case of a more general theorem: if $\varphi$ is an order preserving function from a complete lattice to itself, then there exists a fixed point of $\varphi$; and from this one derives a nice proof of the Shröder-Cantor-Bernstein theorem.

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Also solved by Galen Novello