Problem for 2017 April

Proposed by Dan Jurca

Prove that if $(x_n)_{n=0}^{\infty} \to x$ in a topological space X, then each rearrangement of the terms of (x_n) converges to x; *i.e.*, if $(x_n)_{n=0}^{\infty} \to x$ and $\pi : \mathbf{N} \to \mathbf{N}$ is a bijection, then $(x_{\pi(n)})_{n=0}^{\infty} \to x$.

Solution by the proposer

We must show that if U is an open subset of X and $x \in U$, then $x_{\pi(n)} \notin U$ for (at most) finitely many values of n. So suppose U is open and $x \in U$. Since $(x_n) \to x$, there exists $n_0 \in \mathbf{N}$ such that $n_0 < n \;\Rightarrow\; x_n \in U$. Let $n_1 = \max\{\pi^{-1}(0), \pi^{-1}(1), \pi^{-1}(2), \ldots, \pi^{-1}(n_0)\}$. We show $n_1 < n \;\Rightarrow\; x_{\pi(n)} \in U$.

For if $n_1 < n$ and $x_{\pi(n)} \notin U$, then $\pi(n) \le n_0$. Therefore $\pi(n) = 0$, or $\pi(n) = 1$, or $\pi(n) = 2$, or …, or $\pi(n) = n_0$. Therefore $n = \pi^{-1}(0)$, or $n = \pi^{-1}(1)$, or $n = \pi^{-1}(2)$, or …, or $n = \pi^{-1}(n_0)$. But then $n \in \{\pi^{-1}(0), \pi^{-1}(1), \pi^{-1}(2), \ldots, \pi^{-1}(n_0)\}$, so (by definition of n_1) it follows that $n \le n_1$. This contradicts $n_1 < n$; therefore $n_1 < n \;\Rightarrow\; x_{\pi(n)} \in U$, so $x_{\pi(n)} \in U$ for all but (at most) finitely many values of n, so $(x_{\pi(n)}) \to x$.

Also solved by Winston Teitler and Benjamin Thomas