Problem for 2017 May

Proposed by Dan Jurca

According to Wikipedia an *Egyptian fraction* is a rational number of the form

$$\frac{1}{n_1}+\frac{1}{n_2}+\frac{1}{n_3}+\cdots+\frac{1}{n_k}$$

where the n_i are distinct positive integers. It is well-known and easy to prove that each positive integer (in fact each positive rational number) is expressible as an Egyptian fraction. For example,

$$2=\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{6}.$$

Express 3 as an Egyptian fraction.

Solution by the proposer

Repeatedly using the fact that if n is a positive integer, then

$$\frac{1}{n}=\frac{1}{n+1}+\frac{1}{n(n+1)},$$

and starting with $3=1/1+1/1+1/1$, one finds that

$$3=\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{12}+\frac{1}{13}+\frac{1}{20}+\frac{1}{42}+\frac{1}{43}+\frac{1}{56}+\frac{1}{156}+\frac{1}{1806},$$

and of course other representations are possible.

Also solved by Perry Aliado, who found this nicer expression.

$$3=\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{8}+\frac{1}{9}+\frac{1}{10}+\frac{1}{12}+\frac{1}{18}+\frac{1}{25}+\frac{1}{40}+\frac{1}{100}$$

The proposer also found these.

$$3=\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\frac{1}{10}+\frac{1}{15}+\frac{1}{230}+\frac{1}{57{,}960}$$

$$\begin{aligned}4=&\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\frac{1}{10}+\frac{1}{11}+\frac{1}{12}+\frac{1}{13}+\frac{1}{14}+\frac{1}{15}+\frac{1}{16}+\frac{1}{17}+\frac{1}{18}+\frac{1}{19}+\frac{1}{20}+\\ &\frac{1}{21}+\frac{1}{22}+\frac{1}{23}+\frac{1}{24}+\frac{1}{25}+\frac{1}{26}+\frac{1}{27}+\frac{1}{28}+\frac{1}{29}+\frac{1}{30}+\frac{1}{200}+\frac{1}{77{,}706}+\frac{1}{16{,}532{,}869{,}712}+\\ &\frac{1}{3{,}230{,}579{,}689{,}970{,}657{,}935{,}732}+\frac{1}{36{,}802{,}906{,}522{,}516{,}375{,}115{,}639{,}735{,}990{,}520{,}502{,}954{,}652{,}700}\end{aligned}$$

The proposer conjectures that 3 does not equal the sum of fewer than thirteen distinct unit fractions, and that 4 does not equal the sum of fewer than thirty-five distinct unit fractions.