If \( S = \{ (x_1, y_1), (x_2, y_2), (x_3, y_3), \ldots, (x_n, y_n) \} \) is a set of \( n \) points in the plane, then the center of \( S \) is the point \[
\left( \frac{x_1 + x_2 + x_3 + \cdots + x_n}{n}, \frac{y_1 + y_2 + y_3 + \cdots + y_n}{n} \right).
\]

Suppose \( p(x, y) \) is a polynomial with real coefficients in two variables of degree \( n \), and 
\[ C = \{(x, y) \in \mathbb{R}^2 \mid p(x, y) = 0 \}. \]

Suppose \( \ell_1, \ell_2, \) and \( \ell_3 \) are three parallel lines in the plane, each of which intersects the curve \( C \) in exactly \( n \) points. Prove that the centers of these sets of intersection points lie on a single line; i.e., the centers of \( \ell_1 \cap C, \ell_2 \cap C, \) and \( \ell_3 \cap C \) lie on a line.

Solution by Dan Jurca

First suppose that each of the three parallel lines \( \ell_i \) is horizontal; and suppose an equation of \( \ell_i \) is \( y = y_i \) for some \( y_i \in \mathbb{R} \). There exist polynomials, say \( r_n(y) \) of degree \( < 1 \) and \( r_{n-1}(y) \) of degree \( < 2 \), such that 
\[ p(x, y) = r_n(y)x^n + r_{n-1}(y)x^{n-1} + \text{terms in } x \text{ of lower degree}. \]

For \( i = 1, 2, 3 \) let \( f_i(t) = p(t, y_i) \). Since \( \ell_i \) intersects the curve \( C \) in \( n \) points, say \( (x_{i1}, y_i), (x_{i2}, y_i), \ldots, (x_{in}, y_i) \), it follows that for \( 1 \leq j \leq n \) we have \( f_i(x_{ij}) = p(x_{ij}, y_i) = 0 \). Since \( r_n(y) \) is a real number, and since there exist \( n \) distinct real roots of \( f_i(t) = r_n(y_i)t^n + \cdots \), it follows that the degree of \( f_i(t) \) is at least \( n \). Therefore the degree of \( f_i(t) \) is exactly \( n \), so \( r_n(y_i) \neq 0 \), and it follows that \( r_n(y) \) is a nonzero real constant, say \( r \). Thus there exist \( r \in \mathbb{R}, r \neq 0 \), \( c_1 \in \mathbb{R} \), and \( c_0 \in \mathbb{R} \) such that 
\[ p(x, y) = nx^n + (c_1y + c_0)x^{n-1} + \cdots. \]

If \( S_i = \ell_i \cap C = \{(x_{i1}, y_i), (x_{i2}, y_i), \ldots, (x_{in}, y_i)\} \), the set of intersection points of the line \( \ell_i \) and the curve \( C \), then since \( p(x, y) = 0 \), we recall that for \( i = 1, 2, \) and \( 3 \), and \( j = 1, 2, \ldots, n \): 
\[ f_i(x_{ij}) = 0. \]

Since \( f_i(t) \) is a polynomial of degree \( n \), it follows that the sum of the roots of \( f_i(t) \) equals \(-r_{n-1}(y_i)/r\). Therefore for \( i = 1, 2, \) and \( 3 \) the center of the set \( S_i = \ell_i \cap C \) equals 
\[ \left( \frac{x_{i1} + x_{i2} + x_{i3} + \cdots + x_{in}}{n}, \frac{y_i + y_i + y_i + \cdots + y_i}{n} \right) = \left( \frac{-r_{n-1}(y_i)}{nr}, y_i \right) = \left( \frac{-c_1y_i - c_0}{nr}, y_i \right). \]

These three points are collinear if and only if the determinant of the following matrix equals zero.

\[
\begin{vmatrix}
-c_1y_1 - c_0 \\
-nr \\
-c_1y_2 - c_0 \\
-nr \\
-c_1y_3 - c_0 \\
-nr
\end{vmatrix}
\]

Adding \( c_1/nr \) times column 2 to column 1 one obtains the following matrix with the same determinant.

\[
\begin{vmatrix}
-c_0/nr \\
-nr \\
-c_0/nr \\
-nr \\
-c_0/nr \\
-nr
\end{vmatrix}
\]

and since column 1 equals a multiple of column 3, this matrix is in fact singular. Therefore the centers of \( S_1, S_2, \) and \( S_3 \) are collinear. (In fact these centers lie on the line with equation \( nx + c_1y + c_0 = 0 \).)

Next consider the case that the three lines \( \ell_i \) are vertical, and that an equation of line \( \ell_i \) is \( x = x_i \) for some \( x_i \in \mathbb{R}, i = 1, 2, \) and \( 3 \). Here if \( p(x, y) = s_n(x)y^n + s_{n-1}(x)y^{n-1} + \cdots \), then one shows \( s_n(x) \) is a nonzero constant and \( s_{n-1}(x) \) is of degree \( \leq 1 \), so \( p(x, y) = sy^n + (d_1x + d_0)y^{n-1} + \cdots \). For \( i = 1, 2, \) and \( 3 \) let \( g_i(t) = p(x_i, t) = st^n + (d_1x_i + d_0)t^{n-1} + \cdots \). Then \( g_i(t) = 0 \) iff \( p(x_i, t) = 0 \). If line \( \ell_i \) intersects the curve \( C \) in the points \( (x_i, y_{ij}), j = 1, 2, \ldots, n \), then 
\[ g_i(y_{ij}) = p(x_i, y_{ij}) = 0, \text{ so that, as before, } y_{i1} + y_{i2} + \cdots + y_{in} = -(d_1x_i + d_0)/s. \]

Thus the center of \( \ell_i \cap C \) equals the following point.
\[
\left( \frac{x_i + x_i + \cdots + x_i}{n}, \frac{y_{i1} + y_{i2} + \cdots + y_{in}}{n} \right) = \left( x_i, -\frac{d_1x_i - d_0}{ns} \right)
\]

Again, as before, a certain determinant equals zero, so that these three points (the centers) are collinear.
Before considering the general case — the lines are neither horizontal nor vertical — we look at an example. In the following figure (a) the curve, an ellipse, is the set \( \{(x, y) \in \mathbb{R}^2 \mid p(x, y) = 0\} \), where

\[
p(x, y) = 16x^2 + 9y^2 - 144,
\]

and equations of the parallel lines are \( \ell_1 : y = \frac{1}{\sqrt{3}}x + 3 \), \( \ell_2 : y = \frac{1}{\sqrt{3}}x + 1 \), and \( \ell_3 : y = \frac{1}{\sqrt{3}}x - 2 \).

Thus each of the lines cuts the \( x \)-axis with angle of inclination \( \frac{\pi}{6} \) or \( 30^\circ \). Figure (b) shows the result of rotating figure (a) through an angle of \( -\pi/6 \), or \( -30^\circ \); i.e., \( \pi/6 \) or \( 30^\circ \) clockwise.

Here \( n = 2 \), and the points of intersection (of the lines \( \ell_i \) and the curve \( C \)) and the centers in figure (a) are as follows.

\[
S_1 = \ell_1 \cap C = \left\{ \left( \frac{-9\sqrt{3} \pm 12\sqrt{10}}{19}, \frac{48 \pm 4\sqrt{30}}{19} \right) \right\} \text{ so the center of } S_1 = \left( \frac{-9\sqrt{3}}{19}, \frac{48}{19} \right)
\]

\[
S_2 = \ell_2 \cap C = \left\{ \left( \frac{-3\sqrt{3} \pm 36\sqrt{2}}{19}, \frac{16 \pm 12\sqrt{6}}{19} \right) \right\} \text{ so the center of } S_2 = \left( \frac{-3\sqrt{3}}{19}, \frac{16}{19} \right)
\]

\[
S_3 = \ell_3 \cap C = \left\{ \left( \frac{3\sqrt{3} \pm 12\sqrt{15}}{19}, \frac{-32 \pm 12\sqrt{5}}{19} \right) \right\} \text{ so the center of } S_3 = \left( \frac{6\sqrt{3}}{19}, \frac{-32}{19} \right)
\]

These three centers lie on the line with equation \( y = -\frac{16}{3\sqrt{3}}x \).

Rotating the lines \( \ell_i \) and the curve \( C \) in figure (a) \( 30^\circ \) clockwise results in the horizontal lines \( \ell'_i \) and the curve \( C' \) in figure (b), in which

\[
C' = \{(x, y) \in \mathbb{R}^2 \mid q(x, y) = 0\} \text{ where } q(x, y) = \frac{57x^2 - 14\sqrt{3}xy + 43y^2}{4} - 144,
\]

and equations of the horizontal lines \( \ell'_i \) are \( \ell'_1 : y = 3\sqrt{3}/2 \), \( \ell'_2 : y = \sqrt{3}/2 \), and \( \ell'_3 : y = -\sqrt{3} \). These lines (\( \ell'_1 \), \( \ell'_2 \), and \( \ell'_3 \)) intersect the curve \( C' \) as follows.

\[
S'_1 = \ell'_1 \cap C' = \left\{ \left( \frac{21 \pm 16\sqrt{30}}{38}, \frac{3\sqrt{3}}{2} \right) \right\} \text{ so the center of } S'_1 = \left( \frac{21}{38}, \frac{3\sqrt{3}}{2} \right)
\]

\[
S'_2 = \ell'_2 \cap C' = \left\{ \left( \frac{7 \pm 48\sqrt{6}}{38}, \frac{\sqrt{3}}{2} \right) \right\} \text{ so the center of } S'_2 = \left( \frac{7}{38}, \frac{\sqrt{3}}{2} \right)
\]

\[
S'_3 = \ell'_3 \cap C' = \left\{ \left( \frac{-7 \pm 24\sqrt{5}}{19}, -\sqrt{3} \right) \right\} \text{ so the center of } S'_3 = \left( \frac{-14}{38}, -\sqrt{3} \right)
\]

These three centers lie on the line with equation \( y = \frac{19\sqrt{3}}{7}x \), and this line and the one above — the line with slope \(-16/(3\sqrt{3})\) — intersect at the origin making an angle of \( 30^\circ \).
Consider now the case in which the lines \( \ell_i \) are not vertical; let \( A \) be the following matrix.

\[
A = \begin{bmatrix}
1/\sqrt{m^2 + 1} & m/\sqrt{m^2 + 1} \\
-m/\sqrt{m^2 + 1} & 1/\sqrt{m^2 + 1}
\end{bmatrix}
\] if the slope of each line \( = m \in \mathbb{R} \)

Then for each vector \( \mathbf{x} \) (with initial point at the origin) in the plane, \( A\mathbf{x} \) is the vector \( \mathbf{x}' \) which is \( \mathbf{x} \) rotated \( \tan^{-1} m \) clockwise. That is, the linear transformation \( T_A : \mathbb{R}^2 \to \mathbb{R}^2 \) which (with respect to the standard bases of \( \mathbb{R}^2 \)) is represented by the matrix \( A \) achieves a rotation about the origin of coordinates of \( -\tan^{-1} m \). In the image \( T_A(\mathbb{R}^2) \) the lines \( \ell_i \) have been sent to horizontal lines, say \( \ell'_i \), and the curve \( C \) has been sent to, say, \( C' \). Since the matrix \( A \) is obviously orthogonal (\( A^T A = I_2 \)), \( T_A(\mathbb{R}^2) \) is congruent to \( \mathbb{R}^2 \), and since \( (\text{by the argument above}) \) the centers of \( \ell'_i \cap C' \) are collinear, it follows that so also are the centers of \( \ell_i \cap C \).

Looking again at the previous example we have \( m = 1/\sqrt{3} \), so that in this case

\[
A = \begin{bmatrix}
\sqrt{3}/2 & 1/2 \\
-1/2 & \sqrt{3}/2
\end{bmatrix}.
\]

(One checks that \( A \begin{bmatrix} \sqrt{3} \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \), so that the \( T_A \) in this case does in fact effect a rotation of \( 30^\circ \) clockwise.)

Now we would like a polynomial \( q \) such that \( q(A\mathbf{x}) = 0 \) iff \( p(\mathbf{x}) = 0 \), so we define \( q \) by \( q(\mathbf{x}) = p(A^{-1}\mathbf{x}) \).

Thus, for example, with \( p(\mathbf{x}) = p(x, y) = 16x^2 + 9y^2 - 144 \), we have

\[
q(x, y) = 16 \left( \frac{\sqrt{3}}{2} x - \frac{1}{2} y \right)^2 + 9 \left( \frac{1}{2} x + \frac{\sqrt{3}}{2} y \right)^2 - 144,
\]

\[
= \frac{57x^2 - 14\sqrt{3}xy + 43y^2}{4} - 144
\]
as sketched above. Similarly the rotated lines \( \ell'_i \) are as follows.

\( \ell_1: y = \frac{1}{\sqrt{3}} x + 3 \) becomes \( \frac{1}{2} x + \frac{\sqrt{3}}{2} y = \frac{1}{\sqrt{3}} \left( \frac{\sqrt{3}}{2} x - \frac{1}{2} y \right) + 3 \) or

\[
\frac{1}{2} x + \frac{\sqrt{3}}{2} y = \frac{1}{2} x - \frac{1}{2\sqrt{3}} y + 3,
\]

which simplifies to

\[
y = \frac{3\sqrt{3}}{2},
\]

an equation of a horizontal line \( \ell'_1 \).

Similarly

\( \ell_2: y = \frac{1}{\sqrt{3}} x + 1 \) becomes \( \ell'_2: y = \frac{\sqrt{3}}{2} \)

\( \ell_3: y = \frac{1}{\sqrt{3}} x - 2 \) becomes \( \ell'_3: y = -\sqrt{3} \).

Thus the lines with slope \( 1/\sqrt{3} \) become horizontal lines in the rotated plane.

Finally, if \( C' = \{(x, y) \in \mathbb{R}^2 \mid q(x, y) = 0\} \), then, as shown above, the centers of \( \ell'_i \cap C' \), \( i = 1, 2, \) and \( 3 \), lie on a line; and since \( T_A \) maps \( \mathbb{R}^2 \) onto itself orthogonally, the centers of \( \ell_i \cap C \) also lie on a line, as shown in the sketch.

Remark. One sees that there is nothing special about the number 3 in this problem (except that the assertion is trivial if there are fewer than three lines). That is, if \( p(x, y) \) is a polynomial of degree \( n \) and \( S = \{\ell_\alpha \mid \alpha \in \mathcal{A}\} \) is a set of parallel lines each of which intersects \( C = \{(x, y) \in \mathbb{R}^2 \mid p(x, y) = 0\} \) in exactly \( n \) points, then the centers of \( \ell_\alpha \cap C \), \( \alpha \in \mathcal{A} \), are collinear.