**Problem for 2017 September**

Communicated by Dan Jurca

If $S = \{(x_1, y_1), (x_2, y_2), (x_3, y_3), \ldots, (x_n, y_n)\}$ is a set of $n$ points in the plane, then the *center* of $S$ is the point

$$\left(\frac{x_1 + x_2 + x_3 + \cdots + x_n}{n}, \frac{y_1 + y_2 + y_3 + \cdots + y_n}{n}\right).$$

Suppose $p(x, y)$ is a polynomial with real coefficients in two variables of degree $n$, and

$$C = \{(x, y) \in \mathbf{R}^2 \mid p(x, y) = 0\}.$$

Suppose $\ell_1$, $\ell_2$, and $\ell_3$ are three parallel lines in the plane, each of which intersects the curve $C$ in exactly $n$ points. Prove that the centers of these sets of intersection points lie on a single line; *i.e.*, the centers of $\ell_1 \cap C$, $\ell_2 \cap C$, and $\ell_3 \cap C$ lie on a line.

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Solution by Dan Jurca

First suppose that each of the three parallel lines $\ell_i$ is horizontal; and suppose an equation of $\ell_i$ is $y = y_i$ for some $y_i \in \mathbf{R}$. There exist polynomials, say $r_n(y)$ of degree $< 1$ and $r_{n-1}(y)$ of degree $< 2$, such that $p(x, y) = r_n(y)x^n + r_{n-1}(y)x^{n-1} +$ terms in $x$ of lower degree. For $i = 1,\ 2$, and 3 let $f_i(t) = p(t, y_i)$. Since $\ell_i$ intersects the curve $C$ in $n$ points, say $(x_{i1}, y_i),\ (x_{i2}, y_i),\ \ldots,\ (x_{in}, y_i)$, it follows that for $1 \le j \le n$ we have $f_i(x_{ij}) = p(x_{ij}, y_i) = 0$. Since $r_n(y)$ is a real number, and since there exist $n$ distinct real roots of $f_i(t) = r_n(y_i)t^n + \cdots$, it follows that the degree of $f_i(t)$ is at least $n$. Therefore the degree of $f_i(t)$ is exactly $n$, so $r_n(y_i) \ne 0$, and it follows that $r_n(y)$ is a nonzero real constant, say $r$. Thus there exist $r \in \mathbf{R}$, $r \ne 0$, $c_1 \in \mathbf{R}$, and $c_0 \in \mathbf{R}$ such that $p(x, y) = rx^n + (c_1 y + c_0)x^{n-1} + \cdots$. If $S_i = \ell_i \cap C = \{(x_{i1}, y_i), (x_{i2}, y_i), \ldots, (x_{in}, y_i)\}$, the set of intersection points of the line $\ell_i$ and the curve $C$, then since $p(x_{ij}, y_i) = 0$, we recall that for $i = 1,\ 2$, and 3, and $j = 1,\ 2,\ \ldots, n$: $f_i(x_{ij}) = 0$. Since $f_i(t)$ is a polynomial of degree $n$, it follows that the sum of the roots of $f_i(t)$ equals $-r_{n-1}(y_i)/r$. Therefore for $i = 1,\ 2$, and 3 the center of the set $S_i = \ell_i \cap C$ equals

$$\left(\frac{x_{i1} + x_{i2} + x_{i3} + \cdots + x_{in}}{n}, \frac{y_i + y_i + y_i + \cdots + y_i}{n}\right) = \left(\frac{-r_{n-1}(y_i)}{nr}, y_i\right) = \left(\frac{-c_1 y_i - c_0}{nr}, y_i\right).$$

These three points are collinear if and only if the determinant of the following matrix equals zero.

$$\begin{bmatrix} \dfrac{-c_1 y_1 - c_0}{nr} & y_1 & 1 \\ \dfrac{-c_1 y_2 - c_0}{nr} & y_2 & 1 \\ \dfrac{-c_1 y_3 - c_0}{nr} & y_3 & 1 \end{bmatrix}$$

Adding $c_1/nr$ times column 2 to column 1 one obtains the following matrix with the same determinant.

$$\begin{bmatrix} -c_0/nr & y_1 & 1 \\ -c_0/nr & y_2 & 1 \\ -c_0/nr & y_3 & 1 \end{bmatrix},$$

and since column 1 equals a multiple of column 3, this matrix is in fact singular. Therefore the centers of $S_1$, $S_2$, and $S_3$ are collinear. (In fact these centers lie on the line with equation $nrx + c_1 y + c_0 = 0$.)

Next consider the case that the three lines $\ell_i$ are vertical, and that an equation of line $\ell_i$ is $x = x_i$ for some $x_i \in \mathbf{R}$, $i = 1,\ 2$, and 3. Here if $p(x, y) = s_n(x)y^n + s_{n-1}(x)y^{n-1} + \cdots$, then one shows $s_n(x)$ is a nonzero constant and $s_{n-1}(x)$ is of degree $\le 1$, so $p(x, y) = sy^n + (d_1 x + d_0)y^{n-1} + \cdots$. For $i = 1,\ 2$, and 3 let $g_i(t) = p(x_i, t) = st^n + (d_1 x_i + d_0)t^{n-1} + \cdots$. Then $g_i(t) = 0$ iff $p(x_i, t) = 0$. If line $\ell_i$ intersects the curve $C$ in the points $(x_i, y_{ij})$, $j = 1,\ 2,\ \ldots,\ n$, then $g_i(y_{ij}) = p(x_i, y_{ij}) = 0$, so that, as before, $y_{i1} + y_{i2} + \cdots + y_{in} = -(d_1 x_i + d_0)/s$. Thus the center of $\ell_i \cap C$ equals the following point.

$$\left(\frac{x_i + x_i + \cdots + x_i}{n}, \frac{y_{i1} + y_{i2} + \cdots + y_{in}}{n}\right) = \left(x_i, \frac{-d_1 x_i - d_0}{ns}\right)$$

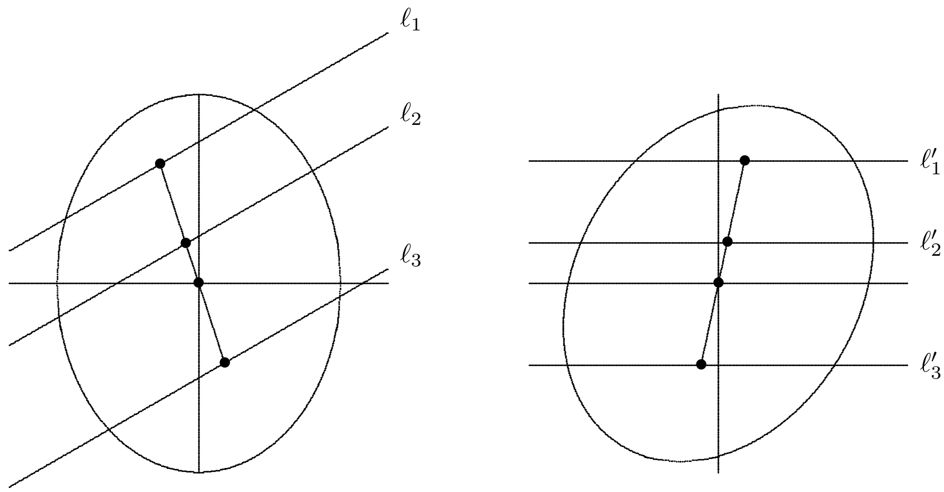
Again, as before, a certain determinant equals zero, so that these three points (the centers) are collinear.

Before considering the general case — the lines are neither horizontal nor vertical — we look at an example. In the following figure $(a)$ the curve, an ellipse, is the set $\{(x, y) \in \mathbf{R^2} \mid p(x, y) = 0\}$, where

$$p(x, y) = 16x^2 + 9y^2 - 144,$$

and equations of the parallel lines are $\ell_1 : \ y = \dfrac{1}{\sqrt{3}}x + 3, \quad \ell_2 : \ y = \dfrac{1}{\sqrt{3}}x + 1,$ and $\ell_3 : \ y = \dfrac{1}{\sqrt{3}}x - 2.$

Thus each of the lines cuts the $x$-axis with angle of inclination $\pi/6$ or $30°$. Figure $(b)$ shows the result of rotating figure $(a)$ through an angle of $-\pi/6$, or $-30°$; *i.e.*, $\pi/6$ or $30°$ clockwise.




$(a)$ $(b)$

Here $n = 2$, and the points of intersection (of the lines $\ell_i$ and the curve $C$) and the centers in figure $(a)$ are as follows.

$$S_1 = \ell_1 \cap C = \left\{ \left( \frac{-9\sqrt{3} \pm 12\sqrt{10}}{19}, \frac{48 \pm 4\sqrt{30}}{19} \right) \right\} \text{ so the center of } S_1 = \left( \frac{-9\sqrt{3}}{19}, \frac{48}{19} \right)$$

$$S_2 = \ell_2 \cap C = \left\{ \left( \frac{-3\sqrt{3} \pm 36\sqrt{2}}{19}, \frac{16 \pm 12\sqrt{6}}{19} \right) \right\} \text{ so the center of } S_2 = \left( \frac{-3\sqrt{3}}{19}, \frac{16}{19} \right)$$

$$S_3 = \ell_3 \cap C = \left\{ \left( \frac{6\sqrt{3} \pm 12\sqrt{15}}{19}, \frac{-32 \pm 12\sqrt{5}}{19} \right) \right\} \text{ so the center of } S_3 = \left( \frac{6\sqrt{3}}{19}, \frac{-32}{19} \right)$$

These three centers lie on the line with equation $y = -\dfrac{16}{3\sqrt{3}}x$.

Rotating the lines $\ell_i$ and the curve $C$ in figure $(a)$ $30°$ clockwise results in the horizontal lines $\ell'_i$ and the curve $C'$ in figure $(b)$, in which

$$C' = \{(x, y) \in \mathbf{R}^2 \mid q(x, y) = 0\} \text{ where } q(x, y) = \frac{57x^2 - 14\sqrt{3}xy + 43y^2}{4} - 144,$$

and equations of the horizontal lines $\ell'_i$ are $\ell'_1 : \ y = 3\sqrt{3}/2, \ \ell'_2 : \ y = \sqrt{3}/2$, and $\ell'_3 : \ y = -\sqrt{3}$. These lines ($\ell'_1$, $\ell'_2$, and $\ell'_3$) intersect the curve $C'$ as follows.

$$S'_1 = \ell'_1 \cap C' = \left\{ \left( \frac{21 \pm 16\sqrt{30}}{38}, \frac{3\sqrt{3}}{2} \right) \right\} \text{ so the center of } S'_1 = \left( \frac{21}{38}, \frac{3\sqrt{3}}{2} \right)$$

$$S'_2 = \ell'_2 \cap C' = \left\{ \left( \frac{7 \pm 48\sqrt{6}}{38}, \frac{\sqrt{3}}{2} \right) \right\} \text{ so the center of } S'_2 = \left( \frac{7}{38}, \frac{\sqrt{3}}{2} \right)$$

$$S'_3 = \ell'_3 \cap C' = \left\{ \left( \frac{-7 \pm 24\sqrt{5}}{19}, -\sqrt{3} \right) \right\} \text{ so the center of } S'_3 = \left( \frac{-14}{38}, -\sqrt{3} \right)$$

These three centers lie on the line with equation $y = \dfrac{19\sqrt{3}}{7}x$, and this line and the one above — the line with slope $-16/(3\sqrt{3})$ — intersect at the origin making an angle of $30°$.

Consider now the case in which the lines $\ell_i$ are not vertical; let $A$ be the following matrix.

$$A = \begin{bmatrix} 1/\sqrt{m^2+1} & m/\sqrt{m^2+1} \\ -m/\sqrt{m^2+1} & 1/\sqrt{m^2+1} \end{bmatrix} \quad \text{if the slope of each line } = m \in \mathbf{R}$$

Then for each vector $\mathbf{x}$ (with initial point at the origin) in the plane, $A\mathbf{x}$ is the vector $\mathbf{x}'$ which is $\mathbf{x}$ rotated $\tan^{-1} m$ clockwise. That is, the linear transformation $T_A : \mathbf{R}^2 \to \mathbf{R}^2$ which (with respect to the standard bases of $\mathbf{R}^2$) is represented by the matrix $A$ achieves a rotation about the origin of coordinates of $-\tan^{-1} m$. In the image $T_A(\mathbf{R}^2)$ the lines $\ell_i$ have been sent to horizontal lines, say $\ell_i'$, and the curve $C$ has been sent to, say, $C'$. Since the matrix $A$ is obviously orthogonal ($A^T A = I_2$), $T_A(\mathbf{R}^2)$ is congruent to $\mathbf{R}^2$, and since (by the argument above) the centers of $\ell_i' \cap C'$ are collinear, it follows that so also are the centers of $\ell_i \cap C$.

Looking again at the previous example we have $m = 1/\sqrt{3}$, so that in this case

$$A = \begin{bmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{bmatrix}.$$

(One checks that $A \begin{bmatrix} \sqrt{3} \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$, so that the $T_A$ in this case does in fact effect a rotation of 30° clockwise.) Now we would like a polynomial $q$ such that $q(A\mathbf{x}) = 0$ iff $p(\mathbf{x}) = 0$, so we define $q$ by $q(\mathbf{x}) = p(A^{-1}\mathbf{x})$. Thus, for example, with $p(\mathbf{x}) = p(x, y) = 16x^2 + 9y^2 - 144$, we have

$$\begin{aligned} q(x,y) &= 16\left(\frac{\sqrt{3}}{2}x - \frac{1}{2}y\right)^2 + 9\left(\frac{1}{2}x + \frac{\sqrt{3}}{2}y\right)^2 - 144 \\ &= \frac{57x^2 - 14\sqrt{3}xy + 43y^2}{4} - 144 \end{aligned},$$

as sketched above. Similarly the rotated lines $\ell_i'$ are as follows.

$$\ell_1 : \ y = \frac{1}{\sqrt{3}}x + 3 \text{ becomes } \frac{1}{2}x + \frac{\sqrt{3}}{2}y = \frac{1}{\sqrt{3}}\left(\frac{\sqrt{3}}{2}x - \frac{1}{2}y\right) + 3 \text{ or}$$

$$\frac{1}{2}x + \frac{\sqrt{3}}{2}y = \frac{1}{2}x - \frac{1}{2\sqrt{3}}y + 3, \text{ which simplifies to}$$

$$y = \frac{3\sqrt{3}}{2}, \text{ an equation of a horizontal line } \ell_1'.$$

Similarly

$$\ell_2 : \ y = \frac{1}{\sqrt{3}}x + 1 \text{ becomes } \ell_2' : \ y = \frac{\sqrt{3}}{2}$$

$$\ell_3 : \ y = \frac{1}{\sqrt{3}}x - 2 \text{ becomes } \ell_3' : \ y = -\sqrt{3}.$$

Thus the lines with slope $1/\sqrt{3}$ become horizontal lines in the rotated plane.

Finally, if $C' = \{(x, y) \in \mathbf{R}^2 \mid q(x, y) = 0\}$, then, as shown above, the centers of $\ell_i' \cap C'$, $i = 1, 2$, and 3, lie on a line; and since $T_A$ maps $\mathbf{R}^2$ onto itself orthogonally, the centers of $\ell_i \cap C$ also lie on a line, as shown in the sketch.

Remark. One sees that there is nothing special about the number 3 in this problem (except that the assertion is trivial if there are fewer than three lines). That is, if $p(x, y)$ is a polynomial of degree $n$ and $S = \{\ell_\alpha \mid \alpha \in \mathcal{A}\}$ is a set of parallel lines each of which intersects $C = \{(x, y) \in \mathbf{R}^2 \mid p(x, y) = 0\}$ in exactly $n$ points, then the centers of $\ell_\alpha \cap C$, $\alpha \in \mathcal{A}$, are collinear.