Problem for 2017 October

Communicated by Dan Jurca

The first problem appeared as a puzzle in a Mensa publication in Chicago several decades ago; the second generalizes the first.

1. A certain house contains one hundred pounds of gold dust. One night elves sneak into the house and steal 1% of the gold dust and leave the rest. On the next night the elves return and steal 2% of the remaining gold dust and leave the rest. On the third night the elves return and steal 3% of the remaining gold dust and leave the rest; and so on, until on the one hundredth night the elves return and steal all of the remaining gold dust. On which night do the elves steal the most gold?

2. For some positive integer n on the first night elves steal $1/n$ of some gold dust and leave the rest; on the second night the elves steal $2/n$ of the remaining gold dust and leave the rest; on the third night the elves steal $3/n$ of the remaining gold dust and leave the rest; and so on, until on the n-th night the elves steal all of the remaining gold dust. On which night do the elves steal the most gold?

Solution by Dan Jurca

We solve problem 2, and then use the reult to solve problem 1. So suppose the original amount equals A_0, the amount taken on the i-th night equals T_i, and the amount left after the i-th night equals A_i. Then

$$1 \le i \le n \;\Rightarrow\; T_i = \frac{i}{n}A_{i-1} \quad \text{and}$$

$$A_i = A_{i-1} - T_i = A_{i-1}\left(1 - \frac{i}{n}\right) = \frac{n-i}{n}A_{i-1}. \quad \text{Hence, inductively,}$$

$$1 \le i \le n \;\Rightarrow\; A_i = A_0 \cdot \frac{\prod_{j=1}^{i}(n-j)}{n^i} \quad \text{and} \quad T_i = A_0 \cdot \frac{i\prod_{j=1}^{i-1}(n-j)}{n^i}.$$

Hence

$$2 \le i \le n \;\Rightarrow\; \frac{T_{i-1}}{T_i} = \frac{n(i-1)}{i(n+1-i)},$$

and it follows that $T_{i-1} < T_i$ if and only if $n(i-1) < i(n+1-i)$; *i.e.*, if and only if $i^2 - i - n < 0$. The function $\varphi : [0,\infty) \to \mathbf{R}$ by $\varphi(x) = x^2 - x - n$ increases on $[1/2, \infty)$, and $\varphi(x) = 0$ if and only if

$$x = \frac{1+\sqrt{4n+1}}{2}.$$

Therefore $i^2 - i - n < 0$ if and only if $i < \left(1+\sqrt{4n+1}\right)/2$. Therefore $\max\{T_i \mid 0 \le i\} = T_m$ where $m = \lfloor \left(1+\sqrt{4n+1}\right)/2 \rfloor$.

In particular, if n equals one hundred, the elves steal the most gold on the tenth night.

Also solved by John Sayer