Department of Mathematics and Computer Science

Comprehensive Examination–Option I

2014 Autumn

Algebra

1. Let $\phi : G \to G'$ be a group homomorphism with kernel K. Prove the following.
   (a) $\phi(G)$, the image of G, is abelian if and only if $xyx^{-1}y^{-1} \in K$ for all $x, y \in G$.
   (b) $\{x \in G \ : \ \phi(x) = \phi(a)\} = Ka$ for each $a \in G$.

2. Prove that each finite integral domain is a field.

3. Let R be a ring with multiplicative identity $1 \neq 0$, and let F be a field. Prove that if $\phi : R \to F$ is a surjective ring homomorphism, then the kernel of ϕ is a maximal ideal in R.

4. Let V be a vector space over the field F, and let $T : V \to V$ be a linear operator on V. Prove that

$$V_0 = \{\mathbf{v} \in V \mid T^k\mathbf{v} = \mathbf{0} \text{ for some integer } k \geq 0\}$$

   is a subspace of V, and if $T^m\mathbf{v} \in V_0$ for some $m \geq 0$, then $\mathbf{v} \in V_0$.

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Complex Analysis

1. Find the image under the transformation

$$w = \frac{z-1}{z+1}$$

of (a) $\{z \in \mathbf{C} \mid |z+2| = 1\}$ and (b) the imaginary axis.

2. Use the method of residues to evaluate

$$\int_0^\infty \frac{x^2\,dx}{(x^2+1)^3}.$$

3. Find the number of zeros, counting multiplicities, of

$$f(z) = z^6 - 5z^4 + z^3 - 2z$$

inside the circle $\{z \in \mathbf{C} \mid |z| = 1\}$, and justify your conclusion.

4. Find all Laurent series expansions of

$$f(z) = \frac{1}{z(1+z^3)}$$

centered at $z_0 = 0$ and their associated regions of convergence.

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Real Analysis

1. Let $f : \mathbf{R} \to \mathbf{R}$.
   (a) Complete the "ε-δ" definition: Definition. f is *continuous* at x_0 if $\ldots$.
   (b) Use the definition to prove that

$$f(x) = \begin{cases} \dfrac{x}{x-4}, & \text{if } x \neq 4 \\ 0, & \text{if } x = 4 \end{cases}$$

   is continuous at $x_0 = 2$.

2. Let

$$f(x) = \sum_{n=1}^{\infty} \frac{10^n x^{2n}}{n!\cosh(nx)} \quad \text{for each } x \in \mathbf{R}.$$

   Prove that f is continuous on $\mathbf{R}$.

3. Prove: If (M, d) is a metric space, $A \subset M$, A is compact, and $f : A \to \mathbf{R}$ is continuous, then f is uniformly continuous on A.

4. Give an example of a function $f : \mathbf{R} \to \mathbf{R}$ which is differentiable at only one point, and prove that your example is correct.

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Topology

1. Let A be a subset of a topological space X. Prove that A is closed if and only if $\mathrm{bdy}(A) \subseteq A$.

2. Let $(\mathbf{R}, \mathcal{T})$ be the space of real numbers with the "half-open interval topology"; *i.e.*, the intervals $[a, b) = \{x \in \mathbf{R} \mid a \leq x < b\}$ form a base for the topology $\mathcal{T}$. Define

$$f : \mathbf{R} \to \mathbf{R} \text{ by } x \in \mathbf{R} \;\Rightarrow\; f(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } 0 \leq x. \end{cases}$$

   Prove that f is continuous.

3. Let X be a path connected topological space. Prove that X is connected.

4. Prove that each closed subspace of a normal space is normal.

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Applied Analysis

1. Determine the solution $y = \phi(t)$ of

$$y'' + 4y = \begin{cases} 8t, & 0 \le t \le \pi \\ 4\sin 2t, & t > \pi \end{cases}$$

satisfying the initial conditions $y(0) = 0$ and $y'(0) = 2$. Assume that y and y' are continuous at $t = \pi$. Show all work.

2. Solve the following system of equations. Sketch the phase plane of the solutions. Determine the behavior of the solutions as $t \to \infty$ for the given initial conditions.

$$\mathbf{x}' = \begin{pmatrix} -2 & 0 \\ 1 & 2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} e^{-t} \\ -e^{-t} \end{pmatrix}, \qquad \mathbf{x}(0) = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$$

3. Let $f : \mathbf{R} \to \mathbf{R}$.
   (a) Complete the "ε-δ" definition: Definition. f is *continuous* at x_0 if . . . .
   (b) Use the definition to prove that

$$f(x) = \begin{cases} \dfrac{x}{x-4}, & \text{if } x \ne 4 \\ 0, & \text{if } x = 4 \end{cases}$$

   is continuous at $x_0 = 2$.

4. Let

$$f(x) = \sum_{n=1}^{\infty} \frac{10^n x^{2n}}{n! \cosh(nx)} \quad \text{for each } x \in \mathbf{R}.$$

Prove that f is continuous on $\mathbf{R}$.

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Comprehensive Examination–Option III

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Numerical Analysis

1. (a) Prove that there exist exactly two positive solutions of the equation
$$\ln x = (x-4)^2 - 1.$$
   (b) Find an approximation β of the smaller solution α such that $|\alpha - \beta| < 10^{-6}$.
   (c) Prove that your approximation β is in fact within 10^{-6} of (the exact) α.

   Note: For this problem you may not use any graphing or rootfinding capabilities of your calculator.

2. Suppose that $f^{(5)}$ is continuous. Show that
$$f'''(x_0) = \frac{-f(x_0-2h) + 2f(x_0-h) - 2f(x_0+h) + f(x_0+2h)}{2h^3} + O(h^2).$$

3. Let A be a $n \times n$ band matrix of the following form.
$$\begin{pmatrix} 2 & 1 & 0 & 0 & 0 & \cdots & 0 \\ b_1 & 2 & 1 & 0 & 0 & \cdots & 0 \\ c_1 & b_2 & 2 & 1 & 0 & \cdots & \vdots \\ 0 & c_2 & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & c_{n-1} & b_{n-2} & 2 & 1 \\ 0 & 0 & \cdots & 0 & c_{n-2} & b_{n-1} & 2 \end{pmatrix}$$
   (a) Write an efficient algorithm (fully exploiting the sparsity pattern) which generates the upper-triangular result of applying Gaussian elimination to A. You may assume that pivoting is unnecessary.
   (b) Let t_n be the total number of floating point arithmetic operations (additions, subtractions, multiplications, and divisions combined) performed when the algorithm in part (a) is executed; write a formula for t_n.

4. Let S be a symmetric positive definite $n \times n$ matrix. For each $\mathbf{x} \in \mathbf{R}^n$ define $\|\mathbf{x}\| = (\mathbf{x}^t S \mathbf{x})^{1/2}$. Prove that $\|\cdot\|$ is a norm.

   Hint. Each symmetric positive definite matrix S has a Cholesky decomposition of the form $S = LL^t$ where L is a lower triangular matrix. Use this fact to prove $\mathbf{x}^t S \mathbf{y} = \mathbf{y}^t S \mathbf{x} \le (\mathbf{x}^t S \mathbf{x})^{1/2} (\mathbf{y}^t S \mathbf{y})^{1/2}$.

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Linear Programming

1. Prove or disprove using the Complementary Slackness Theorem: (1,0,1,0) is an optimal solution of the following problem.

$$\begin{aligned} \text{Minimize} \quad & 5x_1 + 8x_2 + 4x_3 + 2x_4 \\ \text{Subject to} \quad & x_1 + 2x_2 - x_3 + x_4 \geq 0 \\ & 2x_1 + 3x_2 + x_3 - x_4 \geq 3 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

2. Consider the linear programming (LP) problem.

$$\begin{aligned} \text{Maximize} \quad & 10x_1 + 40x_2 + 80x_3 \\ \text{Subject to} \quad & -x_1 - 2x_2 + x_3 \leq 14 \\ & x_1 + 2x_2 + x_3 \leq 26 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

   (a) Use the Simplex method to solve the problem.

   (b) State the dual of this LP problem. Use your results from part (a) to find the optimal values of the dual variables.

   (c) Suppose the coefficient of x_1 in the objective function of LP is changed from 10 to 22. Use techniques of sensitivity analysis to determine the solution of this revised problem.

3. A commodity is to be shipped from three warehouses to four outlets. The weekly supplies, demands (cases), and transportation costs (in cents per case) are as follows.

	Outlet 1	**Outlet 2**	**Outlet 3**	**Outlet 4**	**Supply**
Warehouse 1	12	15	10	25	200
Warehouse 2	15	19	11	30	300
Warehouse 3	21	30	18	40	200
Demand	200	150	200	150	

   Solve this transportation problem, finding the cheapest way to ship the commodity from the warehouses to the outlets.

Linear Programming–continued

4. Consider the following problem.

$$\begin{aligned} \text{Maximize} \quad & 5x_1 + 8x_2 + 9x_3 \\ \text{Subject to} \quad & 2x_1 + x_2 + x_3 \leq 2 \\ & 4x_1 + 2x_2 + 3x_3 \leq 3 \\ & x_1 + 3x_2 + 3x_3 \leq 4 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

Below are the first and last tableaux in the Simplex method solution of this maximization problem.

	x_1	x_2	x_3	x_4	x_5	x_6	
x_4	2	1	1	1	0	0	2
x_5	4	2	3	0	1	0	3
x_6	1	3	3	0	0	1	4
	-5	-8	-9	0	0	0	0
x_4	$\frac{5}{3}$	0	0	1	0	$-\frac{1}{3}$	$\frac{2}{3}$
x_3	$\frac{10}{3}$	0	1	0	1	$-\frac{2}{3}$	$\frac{1}{3}$
x_2	-3	1	0	0	-1	1	1
	1	0	0	0	1	2	11

For each of the following scenarios return to the original problem. Use sensitivity analysis to answer each question.

(a) What is the range on the coefficient of x_2 in the objective function such that the basis variables do not change?

(b) Find the new solution if the constant in the third constraint changes from 4 to 6: $x_1 + 3x_2 + 3x_3 \leq 6$.

(c) Add the constraint $2x_1 + 2x_2 + 3x_3 \leq 2$. What is the new solution?

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Probability

1. The negative binomial distribution is used to measure the number of Bernoulli trials one attempts until the rth success occurs. The probability mass function for the negative binomial is

$$P(X = x) = \binom{x-1}{r-1} p^r (1-p)^{x-r}, \quad \text{where } x = r,\ r+1,\ r+2,\ \ldots\ .$$

   (a) Let Y represent the number of failures, rather that the number of trials, until the rth success. Write the probability mass function for Y.

   (b) Derive the expected value of Y.

   (c) Use the result from part (b) to find the expected value of X.

2. In a medical experiment a rat has been exposed to some radiation. The experimenters believe that the rat's survival time X (in weeks) has the probability density function (p.d.f)

$$f(x) = \frac{3x^2}{120^3} e^{-(x/120)^3}, \qquad 0 < x < \infty.$$

   (a) What is the probability that the rat survives at least 100 weeks?

   (b) What is the probability that the rat survives between 100 weeks and 130 weeks?

   (c) Find the expected value of the survival time.

   (d) Find the median survival time.

   (e) Find the standard deviation of the survival time.

   Hints. You may find the following useful.

   i. The definition: $0 < t \;\Rightarrow\; \Gamma(t) = \int_0^\infty y^{t-1} e^{-y}\, dy$;

   ii. and the fact that if n is a positive integer, then $\Gamma(n) = (n-1)!$.

Probability–continued

3. Suppose that we toss a fair coin until a head first comes up, and let X represent the number of tosses that were made. Then the possible values of X are $1,\ 2,\ \ldots$, and the distribution function of X is defined by

$$m(i) = \frac{1}{2^i}$$

which is just the geometric distribution with parameter 1/2.

   (a) Find the expected value of X. Does this fit your intuition how it should be, given that the coin is fair?

   (b) Suppose that we flip a fair coin until a head first appears, and if the number of tosses equals n, then we are paid 2^n dollars. What is the expected value of the payment?

   (c) From what we learn in (b), how much would you be willing to pay per game for the privilege of playing this game?

4. A medical research team wishes to assess the usefulness of a certain symptom (S) in the diagnosis of a particular disease (D). In a random sample of 775 patients with the disease, 744 reported having the symptom. In an independent random sample of 1380 subjects without the disease, 21 reported that they had the symptom.

   (a) Compute the sensitivity of the symptom, $P(S|D)$.

   (b) Compute the specificity of the symptom, $P(S^c|D^c)$, where ^{c} indicates the complement of the event.

   (c) Suppose it is known that the rate of the disease in the general population is 0.002, $P(D) = 0.002$.

      i. What is the positive predictive value of the symptom, $P(D|S)$?

      ii. What is the negative predictive value of the symptom, $P(D^c|S^c)$?

   (d) Find the positive predictive values for the symptom for the following disease rates: 0.001, 0.01, 0.1.

   (e) What do you conclude about the positive predictive value of the symptom on the basis of the results obtained in part (d)?