Department of Mathematics

Comprehensive Examination–Option I

2016 Autumn

Algebra

1. Prove: Each subgroup of a cyclic group is cyclic.

2. Suppose that $\phi : R \to S$ is a ring homomorphism from a commutative ring R with unity 1 onto a ring S. Prove:
   (a) $\ker \phi$ is an ideal of R.
   (b) S is a commutative ring with unity.

3. Let $\mathbf{Q}[x]$ be the ring of polynomials with coefficients in $\mathbf{Q}$, the field of all rational numbers. Prove or disprove:
   (a) $(x^3 + 4x^2 + 3x + 2)$ is a maximal ideal of $\mathbf{Q}[x]$.
   (b) $\mathbf{Q}[x]/(x^3 + 4x^2 + 3x + 2)$ is a field.

4. Let A be a real $n \times n$ matrix and let A^T be its transpose matrix. Prove that $A^T A$ is invertible if and only if the column vectors of A are linearly independent

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Complex Analysis

1.

2.

3.

4.

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Real Analysis

1. Let $A = \{\sqrt{q} \mid q \in \mathbf{Q} \text{ and } 0 \leq q\}$. Prove that A has zero measure (with the standard measure of Euclidean space).

   Hint: Use the fact that the measure of a countable disjoint union of sets equals the sum of their measures.

2. State and prove Rolle's theorem.

3. Let
$$f(x) = \sum_{n=1}^{\infty} \frac{1}{n^2 x + 1}.$$
   Prove that for each $a,\ a > 0$, f is continuous on the interval $[a, \infty)$.

4. A function $f : \mathbf{R} \to \mathbf{R}$ if continuous if any of the following four equivalent conditions holds.

   (i) For each $x_0 \in \mathbf{R}$, if $\varepsilon > 0$ then $\exists\, \delta$ such that $\delta > 0$ and $|x_0 - x| < \delta \Rightarrow |f(x_0) - f(x)| < \varepsilon$.

   (ii) For each $x_0 \in \mathbf{R}$, if the sequence (x_i) converges to x_0, then $(f(x_i)) \to f(x_0)$.

   (iii) If U is an open subset of $\mathbf{R}$, then $f^{-1}(U)$ is an open subset of $\mathbf{R}$.

   (iv) If F is a closed subset of $\mathbf{R}$, then $f^{-1}(F)$ is a closed subset of $\mathbf{R}$.

   Let $f(x) = \begin{cases} 3 - x, & x \leq 1 \\ x - 1, & x > 1 \end{cases}$. Use two of the above four characterizations of continuity to prove that f is not continuous.

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Topology

1. Prove: Each metric space is Hausdorff.

2. Suppose X and Y are topological spaces, $f : X \to Y$ is continuous, and A is a compact subset of X. Prove that $f(A)$ is compact.

3. Let X be a topological space. Prove: If A is a connected subset of X and B is a subset of X such that $A \subseteq B \subseteq \overline{A}$, then B is conneccted.

4. Suppose A, X, and Y are topological spaces, $f_1 : A \to X$ and $f_2 : A \to Y$ are continuous functions, and $f : A \to X \times Y$ by $f(a) = (f_1(a), f_2(a))\ \forall a \in A$, where $X \times Y$ has the usual product topology. Prove that f is continuous.

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Applied Analysis

1. Find the solution about the point $x = 0$ of the following initial value problem. You need to show only the values through the x^4 term.

$$(x+2)y'' + xy' - 3y = 0, \qquad y(0) = 3, \quad y'(0) = -5$$

2. Solve the following system of equations.

$$\mathbf{x}' = \begin{bmatrix} 1 & 2 & 2 \\ 4 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix} \mathbf{x}, \qquad \mathbf{x}(0) = \begin{bmatrix} 10 \\ 11 \\ 6 \end{bmatrix}$$

3. Let

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{n^2 x + 1}.$$

Prove that for each $a,\ a > 0$, f is continuous on the interval $[a, \infty)$.

4. A function $f : \mathbf{R} \to \mathbf{R}$ if continuous if any of the following four equivalent conditions holds.
   (i) For each $x_0 \in \mathbf{R}$, if $\varepsilon > 0$ then $\exists\, \delta$ such that $\delta > 0$ and $|x_0 - x| < \delta \ \Rightarrow\ |f(x_0) - f(x)| < \varepsilon$.
   (ii) For each $x_0 \in \mathbf{R}$, if the sequence (x_i) converges to x_0, then $(f(x_i)) \to f(x_0)$.
   (iii) If U is an open subset of $\mathbf{R}$, then $f^{-1}(U)$ is an open subset of $\mathbf{R}$.
   (iv) If F is a closed subset of $\mathbf{R}$, then $f^{-1}(F)$ is a closed subset of $\mathbf{R}$.

   Let $f(x) = \begin{cases} 3 - x, & x \leq 1 \\ x - 1, & x > 1 \end{cases}$. Use two of the above four characterizations of continuity to prove that f is not continuous.

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Numerical Analysis

1. (a) Prove that there exists exactly one solution of the equation $\arctan x = e^{-x}$.
   (b) Use Newton's method to find an approximation β of the solution α such that $|\alpha - \beta| < 10^{-6}$.
   (c) Prove that your approximation is in fact within 10^{-6} of (the exact) α.
   Note: For this problem you may not use any graphing or root finding capabilities of your calculator.

2. Consider the following difference formula.

$$f'(x_0) = \frac{f(x_0+4h) - 12f(x_0+2h) + 32f(x_0+h) - 21f(x_0)}{12h} - f^{(4)}(x_0)h^3 + O(h^4)$$

   Note that $\dfrac{f(x_0+4h) - 12f(x_0+2h) + 32f(x_0+h) - 21f(x_0)}{12h}$ is a $O(h^3)$ approximation of $f'(x_0)$.
   Use the formula to construct a $O(h^4)$ approximation of $f'(x_0)$ that involves

$$f(x_0),\ f(x_0+h),\ f(x_0+2h),\ f(x_0+4h),\ \text{and } f(x_0+8h).$$

3. Let A be a symmetric pentadiagonal positive definite matrix of the following form.

$$\begin{pmatrix}
a_1 & 1 & b_1 & 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\
1 & a_2 & 0 & b_2 & 0 & & & & & & \cdot \\
b_1 & 0 & a_3 & 0 & b_3 & 0 & & & & & \cdot \\
0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & & & \cdot \\
\cdot & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & & \cdot \\
\cdot & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot \\
\cdot & & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & \cdot \\
\cdot & & & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\
\cdot & & & & & 0 & b_{n-4} & 0 & a_{n-2} & 0 & b_{n-2} \\
\cdot & & & & & & 0 & b_{n-3} & 0 & a_{n-1} & 0 \\
0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & b_{n-2} & 0 & a_n
\end{pmatrix}$$

   Write an algorithm which generates a lower-triangular L such that $A = LL^t$. That is, construct the vectors $L0$, $L1$, and $L2$ such that

$$L = \begin{pmatrix}
L0_1 & 0 & 0 & \cdot & \cdot & \cdot & 0 \\
L1_1 & L0_2 & 0 & & & & \cdot \\
L2_1 & L1_2 & L0_3 & 0 & & & \cdot \\
0 & \cdot & \cdot & \cdot & \cdot & & \cdot \\
\cdot & 0 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & & 0 & \cdot & \cdot & \cdot & 0 \\
0 & \cdot & \cdot & 0 & L2_{n-2} & L1_{n-1} & L0_n
\end{pmatrix}$$

   Use the given single subscripting and fully exploit the sparsity pattern. Assume that $n \geq 3$.

4. (a) Define *strictly diagonally dominant* $n \times n$ *matrix*.
   (b) Let A be a strictly diagonally dominant $n \times n$ matrix, and let T_j be its Jacobi iteration matrix. Prove that $\|T_j\|_\infty < 1$.

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Linear Programming

1. Solve the following linear programming problem using the Primal Simplex Method (NOT the Dual Method).

$$\begin{aligned} \text{minimize} \quad & 6x_1 + 3x_2 + 4x_3 \\ \text{subject to} \quad & 3x_1 + x_2 + 3x_3 \geq 9 \\ & 2x_1 - x_2 + 4x_3 \leq 8 \\ & 2x_1 + 2x_2 + x_3 \geq 9 \\ & x_j \geq 0 \end{aligned}$$

2. Consider the following maximization problem.

$$\begin{aligned} \text{maximize} \quad & 40x_1 + 20x_2 + 10x_3 \\ \text{subject to} \quad & 3x_1 + x_2 + 4x_3 \leq 18 \\ & 2x_1 + 2x_2 + x_3 \leq 16 \\ & x_1 + x_2 + 3x_3 = 14 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

The beginning and final tableaux in the Simplex method are given in the following table.

	x_1	x_2	x_3	x_4	x_5	a_1	
x_4	3	1	4	1	0	0	18
x_5	2	2	1	0	1	0	16
a_1	1	1	3	0	0	1	14
	-40	-20	-10	0	0	0	0
x_3	0	0	1	0	$-1/5$	$2/5$	$12/5$
x_1	1	0	0	$1/2$	$1/10$	$-7/10$	$4/5$
x_2	0	1	0	$-1/2$	$1/2$	$1/2$	6
	0	0	0	10	12	-14	176

For each of the following scenarios return to the original problem. Use sensitivity analysis to answer the questions.

(a) How much can the coefficient of x_2 change in the objective function and still have the same values for the solution set?

(b) Suppose the right hand side constant in the last constraint changes from 14 to 7. Does the solution change? If so, find the new solution.

(c) Suppose another constraint is added:

$$4x_1 + 2x_2 + x_3 \leq 16$$

How does this affect the solution? If the solution changes, what is the new solution?

Linear Programming — continued

3. Using the Complementary Slackness Theorem prove or disprove the following statement: $(4,2,0)$ is the optimal solution of the maximization problem below.

$$\begin{array}{rrcl} \text{maximize} & 5x_1 + 2x_2 + \ \ x_3 & & \\ \text{subject to} & x_1 + 2x_2 + 2x_3 & \leq & 14 \\ & 2x_1 + 3x_2 + 4x_3 & \leq & 14 \\ & x_1 + 2x_2 + \ \ x_3 & = & 8 \\ & x_j & \geq & 0 \end{array}$$

4. Solve the following transportation problem, with transportation costs given inside the table. The supplies are listed along the left, and the demands are listed along the top.
Make sure to give the final minimum cost.

	30	20	25	30	25
20	7	11	10	9	8
30	7	3	2	4	5
30	6	6	9	4	5
50	8	10	12	11	9

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Probability

1. A coin is biased so that heads is twice as likely as tails. For three independent tosses of the coin find
   (a) the probability distribution $f(x)$ of X, the total number of heads;
   (b) the probability of getting at most two heads;
   (c) the distribution function $F(X)$ for X;
   (d) $P(X > 2)$ using part (c);
   (e) the expected number of heads, $E(X)$.

2. Let X be a Gamma random variable with parameters a and b, where $a > 0$ and $b > 0$. The Gamma probability density function is

$$f(x) = \frac{1}{\Gamma(a)b^a} x^{a-1} e^{-x/b}, \quad 0 < x < \infty.$$

   (a) Show that the moment generating function of X is

$$M(t) = \left(\frac{1}{1-bt}\right)^a, \quad t < 1/b.$$

   (b) Find the moment generating function of $Y = 2X/b$.
   (c) What does the uniqueness property of moment generating function tell you about the distribution of Y? What is the probability density function of Y?

3. Let

$$f(x) = \begin{cases} x^2 & \text{if } 0 < x < 1 \\ k(x-1) & \text{if } 1 \le x < 3 \\ 1/12 & \text{if } 3 \le x < 5 \\ 0 & \text{otherwise.} \end{cases}$$

   (a) Find the constant k so that $f(x)$ is a probability density function.
   (b) Find the cumulative distribution function of X.
   (c) Find the expected value and variance of X, $E(X)$, $Var(X)$.
   (d) Find the median of X. That is, find a value m such that $P(X < m) = 0.5$.

4. Let the sample space Ω be the collection of all five-card hands from a deck of 52 cards, and let each hand in Ω be equally likely. Let

$$B_i = \{\omega \mid \omega \text{ has in it } i \text{ aces}\}$$

   for $i = 0, 1, 2, 3, 4$, and

$$A = \{\omega \mid \omega \text{ has in it three black cards}\}.$$

   (a) How many five-card hands are possible?
   (b) Find the prior probabilities of B_0, B_1, B_2, B_3, and B_4. Keep four decimal places in the calculations.
   (c) Find the posterior probability B_i given A, for $i = 0,\ 1,\ 2,\ 3$, and 4. Keep four decimal places in the calculations.
   (Hint: Compute the conditional probabilities directly. Using Bayes' Rule is not necessary.)
   (d) For which of B_0, B_1, B_2, B_3, and B_4 is the posterior probability higher than the prior probability?