Department of Mathematics
Comprehensive Examination
2019 Fall Semester Part 1: Core Classes

Directions:

- You will answer FOUR questions.
- You MUST choose at least ONE question from each class (one from Math 620, one from Math 630, and one from Math 670).
- The fourth question can come from any of the classes.

Time: 2.5 hours

Math 620

1. Let G be a group. Prove that G is abelian if and only if the map $\phi : G \to G$ defined via $\phi(g) = g^{-1}$ is a group automorphism.

2. Let R be an integral domain. Prove that for a in R, $\langle a \rangle = R$ if and only if a is a unit.

Math 630

3. Consider the sequence of functions $f_n(x) = x^{2n}$.
   (a) Prove that the sequence converges uniformly on $\left[0, \frac{1}{2}\right]$.
   (b) Prove that the sequence of functions does NOT converge uniformly on $[0, 1]$.

4. Let $M = \{m, a, t, h\}$ and let d be the discrete metric.
   (a) Prove, directly from the definition, that $A = \{a\}$ is open in M.
   (b) Prove that (M, d) is complete.

Please turn over $\Longrightarrow$

5. Consider the equation $e^x = 3 - (x-1)^2$
   (a) Prove that the equation has exactly two solutions.
   (b) Choose of the of two solutions. Use Newton's Method to find an approximation with an absolute error of less than 10^{-6}. Note: for this problem, you may not use any graphing or root finding capabilities on your calculator.

6. Given that the eigenvalues of the 1-dimensional Laplace matrix

$$\begin{pmatrix} 2 & -1 & 0 & 0 & \dots & \dots & 0 \\ -1 & 2 & -1 & 0 & \dots & \dots & 0 \\ 0 & \ddots & \ddots & \ddots & 0 & \dots & 0 \\ 0 & 0 & . & . & . & 0 & 0 \\ \vdots & & & \ddots & \ddots & \ddots & 0 \\ \vdots & & & 0 & -1 & 2 & -1 \\ 0 & & & 0 & 0 & -1 & 2 \end{pmatrix}$$

are all on the interval $(0, 4)$,

   (a) Prove that $\rho(FE) < 1$ for $0 < \mu < \frac{1}{2}$ where FE is the tridiagonal Forward Euler matrix

$$\begin{pmatrix} 1-2\mu & \mu & & & \\ \mu & 1-2\mu & \mu & & \\ & \ddots & \ddots & \ddots & \\ & & \mu & 1-2\mu & \mu \\ & & & \mu & 1-2\mu \end{pmatrix}$$

   (b) Prove that $\rho((BE)^{-1}) < 1$ for all $\mu > 0$ where BE is the tridiagonal Backward Euler matrix

$$\begin{pmatrix} 1+2\mu & -\mu & & & \\ -\mu & 1+2\mu & -\mu & & \\ & \ddots & \ddots & \ddots & \\ & & -\mu & 1+2\mu & -\mu \\ & & & -\mu & 1+2\mu \end{pmatrix}$$

Note that $\rho(A)$ is the spectral radius of A.

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2019 Fall Semester Part 2: "Choose 2" Classes

Directions: You will answer THREE questions from a total of four questions, posed from two classes.

Time: 2 hours

Math 640: Complex Analysis

1. Let D be an open connected subset of $\mathbb{C}$ and let $f : D \to \mathbb{C}$ be analytic in D
   Prove that if $\overline{f}$ is analytic in D then f is constant in D.

2. Prove: If f is analytic within and on a simple closed contour Γ, and z_0 is not on Γ, then

$$\int_{z \in \Gamma} \frac{f(z)}{(z - z_0)^2} \mathrm{d}z = \int_{z \in \Gamma} \frac{f'(z)}{z - z_0} \mathrm{d}z$$

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Time: 2 hours

Math 660: Topology

3. Let X be a Hausdorff space and $f: X \to X$ be a continuous function.
   Prove: $F = \{x \in X : f(x) = x\}$ is closed in X.

4. Let A and B be disjoint compact subspaces of a Hausdorff space X.
   Prove: There exist disjoint open subsets U and V of X such that $A \subseteq U$ and $B \subseteq V$.

   (Recall: For any compact subspace Y of a Hausdorff space X and each point x_0 of X not in Y, there exist disjoint open neighborhoods of x_0 and Y.)

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Math 675: Differential Equations

5. Consider the following differential equation

$$(x^2 - 1)y'' + xy' - y = 0$$

   (a) Find the recursion relation for the series solution centered at $x_0 = 0$.

   (b) Find the first six non-zero terms of the series solution centered at $x_0 = 0$. Write your final answer in terms of the coefficients a_0 and a_1.

6. Consider the system of differential equations given by

$$\mathbf{x}' = \begin{pmatrix} 0 & 2 \\ -1 & 3 \end{pmatrix} \mathbf{x}$$

   (a) What is the general solution to the homogeneous differential equation above?

   (b) What is the general solution to the inhomogeneous system:

$$\mathbf{x}' = \begin{pmatrix} 0 & 2 \\ -1 & 3 \end{pmatrix} \mathbf{x} + \begin{pmatrix} e^t \\ -e^t \end{pmatrix}.$$

   Simplify your answer.

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Math 680: Optimization

7. Solve the following problem using the Simplex method. Clearly show the set up of the problem, the Simplex tableau, and the solution.

$$\begin{array}{rrrrl} \text{maximize} & 6x_1 & + 9x_2 & + 10x_3 & \\ \text{subject to} & 3x_1 & + 2x_2 & - 6x_3 & \leq 24 \\ & x_1 & + 5x_2 & + 2x_3 & \leq 18 \\ & 3x_1 & + 3x_2 & + 4x_3 & = 24 \\ & & & x_1, x_2, x_3 & \geq 0 \end{array}$$

8. Consider the problem

$$\begin{array}{rrrrl} \text{maximize} & 4x_1 & + 8x_2 & + 3x_3 & \\ \text{subject to} & 2x_1 & - 3x_2 & + 2x_3 & \leq 26 \\ & -3x_1 & + x_2 & + 4x_3 & \leq 24 \\ & 3x_1 & + 2x_2 & - 2x_3 & \leq 30 \\ & & & x_1, x_2, x_3 & \geq 0 \end{array}$$

The first and last tableau are shown below.

	x_1	x_2	x_3	x_4	x_5	x_6	b
x_4	2	-3	2	1	0	0	26
x_5	-3	1	4	0	1	0	24
x_6	3	2	-2	0	0	1	30
	-4	-8	-3	0	0	0	0
x_1	1	0	0	$\frac{5}{28}$	$\frac{1}{28}$	$\frac{1}{4}$	13
x_3	0	0	1	$\frac{9}{56}$	$\frac{13}{56}$	$\frac{1}{8}$	$\frac{27}{2}$
x_2	0	1	0	$-\frac{3}{28}$	$\frac{5}{28}$	$\frac{1}{4}$	9
	0	0	0	$\frac{19}{56}$	$\frac{127}{56}$	$\frac{27}{8}$	$\frac{329}{2}$

Use sensitivity analysis to answer the questions below. For each situation, return to the original problem as given.

(a) How much can c_2, the coefficient of x_2, change in the objective function and not change the solution of $(x_1, x_2, x_3) = \left(13, 9, \frac{27}{2}\right)$?

(b) What would be the new solution if the following constraint is added to the system?

$$x_1 - 2x_2 + 3x_3 \leq 12$$