Department of Mathematics
Comprehensive Examination
2020 Fall Semester Part 1: Core Classes

Time: 2.5 hours **Directions:**

- You will answer FOUR questions.
- You must choose at least ONE question from each class (one from Math 620, one from Math 630, and one from Math 670).
- The fourth question can come from any of the classes.

Math 620

1. Let p and q be distinct primes. Suppose G is a group of order pq such that G has a normal subgroup P of order p and a normal subgroup Q of order q. Prove:
   (a) $P \bigcap Q = \langle e \rangle$, where e is the identity of G.
   (b) $xy = yx \ \forall x \in P$ and $\forall y \in Q$.
   (c) G is cyclic.
2. Let R be a ring with multiplicative identity $1 \neq 0$ and let F be a field. Prove:
   (a) The only ideals of F are $\{0\}$ and F itself.
   (b) If $\phi \colon R \to F$ is a surjective ring homomorphism, then the kernel K of ϕ is a maximal ideal in R.

Math 630

3. Let $\langle \mathbb{R}, \rho \rangle$ be the set $\mathbb{R}$ with the Euclidean metric. Let $x_0 \in \langle \mathbb{R}, \rho \rangle$ and $\{x_n\}_{n=1}^{\infty}$ be a sequence in $\langle \mathbb{R}, \rho \rangle$ such that $x_n \to x_0$ as $n \to \infty$. Suppose that $x_0 > 0$.
   (a) Prove that the terms x_n can be equal to zero at most a finite number of times.
   (b) Prove, directly from the definition, that there exists a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ satisfying the following two properties:
      i. $x_{n_k} \neq 0$ for all positive integers k, and,
      ii. $\lim\limits_{k\to\infty} \frac{1}{x_{n_k}} = \frac{1}{x_0}$.
4. Let $0 < a < b$ be fixed real numbers. Let
$$f(x) = \sum_{n=1}^{\infty} \frac{nx}{n^3 x + 1}.$$
   Prove that f is continuous on the interval (a, b).

Math 670

5. Consider $x^2 = \cos x$
   (a) Prove that the equation has exactly two solutions.
   (b) Choose one of the two solutions. Use Newton's Method to find an approximation with an absolute error of less than 10^{-5}. Note: for this problem, you may not use any graphing or root finding capabilities on your calculator.
6. Let $A = LL^T$ where A is nonsingular and symmetric and L is lower-triangular. Prove that A is positive definite.

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2020 Fall Semester Part 2: "Choose 2" Classes

Directions: You will answer THREE questions from a total of four questions, posed from two classes.

Time: 2 hours

Math 640: Complex Analysis

1. Find a Linear Fractional Transformation that maps the unit disk to the upper half plane.

2. Find the Laurent Series expansion of

$$f(z) = z^3 \cos\left(\frac{1}{z^2}\right)$$

about the point $z_0 = 0$, determine the largest open set in which the series converges, and evaluate

$$\int_C f(z)dz$$

where C is the circle of radius $\frac{1}{4}$ centered at 0 and traversed once counterclockwise.

Math 675: Differential Equations

3. (a) Find the general solution for the system

$$\mathbf{x}'(t) = \begin{pmatrix} -1 & 4 \\ -4 & 7 \end{pmatrix} \mathbf{x}(t) \ .$$

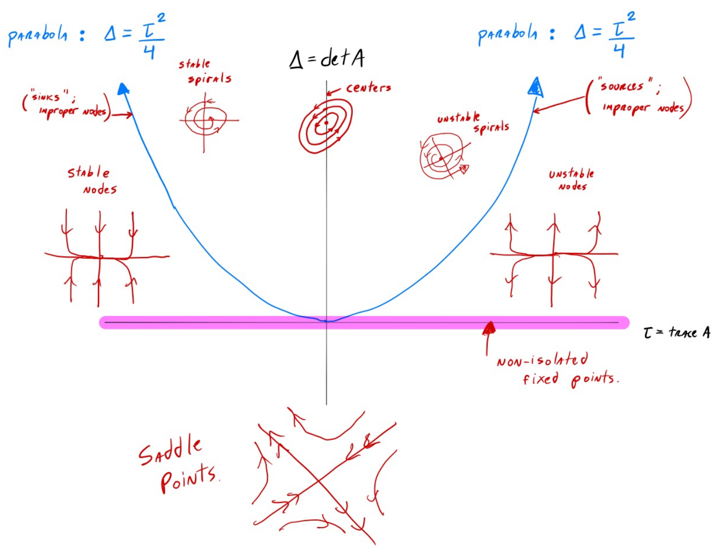
Write your general solution in the form $\mathbf{x}(t) = c_1\mathbf{x}^{(1)}(t) + c_2\mathbf{x}^{(2)}(t)$.

(b) Let $\mathbf{x}^{(1)}(t)$ and $\mathbf{x}^{(2)}(t)$ be as in part (a). Prove that the set $\mathcal{S}(t) = \{\mathbf{x}^{(1)}(t),\ \mathbf{x}^{(2)}(t)\}$ is linearly independent on $(-\infty, \infty)$

(c) What's the behavior of solutions $\mathbf{x}(t)$ as $t \to +\infty$?
Note:

i. You must cover all possible cases.
ii. If there's a "dominant direction line" you must specify whether solutions asymptote to this line or whether they become parallel to this line on the phase plane.

4. Consider the sketch of the trace-determinant plane below



The trace-determinant Plane

Let $0 \le \theta \le \pi$ be a parameter. Consider the system:

$$\mathbf{x}'(t) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \mathbf{x}(t)$$

(a) Sketch the curve in the trace-determinant plane given by

$$\{(\tau, \Delta) | \tau = \text{trace } A, \Delta = \det A, 0 \le \theta \le \pi\}$$

(b) Use part (a) to identify the values of θ where the type of the system changes (i.e. from center to spiral, etc).

Math 680: Optimization

5. Solve the following problem using the Simplex method. Clearly show the set up of the problem, the Simplex tableau, and the solution.

$$\begin{array}{rrrrrl} \text{minimize} & 8x_1 & +\ 10x_2 & +\ 5x_3 & +\ 12x_4 & \\ \text{subject to} & 4x_1 & +\ \ 3x_2 & +\ 6x_3 & -\ \ 2x_4 & \le 30 \\ & -x_1 & +\ \ 2x_2 & -\ 3x_3 & +\ \ 6x_4 & \le 32 \\ & 2x_1 & -\ \ 2x_2 & +\ 2x_3 & +\ \ 3x_4 & = 20 \\ & & & & x_1, x_2, x_3, x_4 & \ge 0 \end{array}$$

6. A Silicon Valley firm specializes in making four types of silicon chips for personal computers. Each chip must go through four stages of processing before completion. First the basic silicon wafers are manufactured, second the wafers are laser etched with a micro circuit, next the circuit is laminated onto the chip, and finally the chip is tested and packaged for shipping. The production manager desires to maximize profits during

the next month, while keeping within the limits of the raw materials, and using all the available testing. The associated maximization problem is

$$\begin{array}{rrrrrl} \text{maximize} & 2000x_1 & +\ 3000x_2 & +\ 5000x_3 & +\ 4000x_4 & \\ \text{subject to} & 100x_1 & +\ 100x_2 & +\ 100x_3 & +\ 100x_4 & \leq 4000 \text{ (raw wafers)} \\ & 10x_1 & +\ 10x_2 & +\ 20x_3 & +\ 20x_4 & \leq 600 \text{ (etching)} \\ & 20x_1 & +\ 20x_2 & +\ 30x_3 & +\ 20x_4 & \leq 900 \text{ (lamination)} \\ & 20x_1 & +\ 10x_2 & +\ 30x_3 & +\ 30x_4 & = 700 \text{ (testing)} \\ & & & & x_1, x_2, x_3, x_4 & \geq 0 \end{array}$$

where x_1, x_2, x_3, x_4 represent the number of 100 chip batches of the four types of chips. Using the Simplex method, the first and last tableaux are shown here.

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	b
x_5	100	100	100	100	1	0	0	4000
x_6	10	10	20	20	0	1	0	600
x_7	20	20	30	20	0	0	1	900
a_1	20	10	30	30	0	0	0	700
	-2000	-3000	-5000	-4000	0	0	0	0
x_2	$\frac{1}{2}$	1	0	0	$\frac{3}{200}$	0	0	25
x_6	-5	0	0	0	$-\frac{1}{20}$	1	0	50
x_3	0	0	1	0	$-\frac{1}{50}$	0	$\frac{1}{10}$	10
x_4	$\frac{1}{2}$	0	0	1	$\frac{3}{200}$	0	$-\frac{1}{10}$	5
	1500	0	0	0	5	0	100	145000

Answer the following questions using Sensitivity Analysis. For each question, refer back to the original problem.

(a) The profit function is a reflection of the profit from each type of silicon chip. Suppose some errors were discovered and the cost function should be

$$\text{maximize } 1800x_1 + 3100x_2 + 4500x_3 + 4300x_4$$

How is the solution affected by this? Thoroughly explain your answer.

(b) The number of raw wafers fluctuates with the markets. Give a range on the value of the right hand side constant, 4000, in the first constraint, such that the solution to the original problem would not change. Namely, the same mix of types of silicon chips would be produced.

(c) Suppose the etching time dips to 540. How does this affect the solution? Thoroughly explain your answer.