Department of Mathematics
Comprehensive Examination
2021 Fall Semester Part 1: Core Classes

Time: 2.5 hours **Directions:**

- You will answer FOUR questions.
- You must choose at least ONE question from each class (one from Math 620, one from Math 630, and one from Math 670).
- The fourth question can come from any of the classes.

Math 620

1.
   (a) Prove that every group of prime order is cyclic.
   (b) Let p be a fixed prime and let G_1 and G_2 be two groups of order p. Prove that G_1 and G_2 are isomorphic.

2.
   (a) Let R be a ring with unity $1 \neq 0$. Give a complete and precise definition that R is an integral domain.
   (b) Let R be a commutative ring with unity $1 \neq 0$ and let P be an ideal in R. Give a complete and precise definition that P is a PRIME ideal.
   (c) Let R be a commutative ring with unity $1 \neq 0$ and let P be a prime ideal in R. Prove that R/P is an integral domain.

Math 630

3. Let $f : M_1 \to M_2$ be a function between two metric spaces (M_1, d_1) and (M_2, d_2). Show that the $\epsilon - \delta$ definition of continuity implies the following property: if $U \subseteq M_2$ is open, then $f^{-1}(U) \subseteq M_1$ is open.

4. Let $f \in \Re[a, b]$ and $a < c < b$. Show that $f \in \Re[a, c]$, $f \in \Re[c, b]$, and
$$\int_a^b f = \int_a^c f + \int_c^b f.$$
   Here $\Re[x, y]$ is the set of Riemann integrable functions $f : [x, y] \to \mathbb{R}$.

Continued on next page.

Math 670

5. Consider the equation

$$ln(x) = (x - 4)^2 - 1$$

   (a) Prove that the equation has exactly two positive solutions.

   (b) Let α be the smaller of the two solutions. Use Newton's Method to find an approximation of α with an absolute error of less than 10^{-5}.

   Note: for this problem, you may not use any graphing or root finding capabilities on your calculator.

6. Let A be an $n \times n$ matrix.

   (a) Complete the definition:

   Let A be an $n \times n$ matrix. Then A is *strictly diagonally dominant* if...

   (b) Let A be strictly diagonally dominant and let T_J be its Jacobi iteration matrix. Prove that $\rho(A) < 1$ where $\rho(A)$ denotes the spectral radius of A.

Department of Mathematics
Comprehensive Examination
2021 Fall Semester Part 2: "Choose 2" Classes

Directions: You will answer THREE questions from a total of four questions, posed from two classes.

Time: 2 hours

Math 660: Topology

1. Recall the following:

   Definition: Let A be a subset of a topological space X. Then A is **dense** in X if and only if $\overline{A} = X$ (i.e. the closure of A is X).

   Prove that A is dense in X if and only if every non-empty open set of X contains a point of A.

2. Let $\mathbb{R}_{\text{har}}$ be the set $\mathbb{R}$ equipped with a topology whose basis consists of all sets of the form (a, b) or $(a, b) - H$ where $H = \{1/n\}_{n\in\mathbb{N}}$ is the harmonic sequence, and $a, b \in \mathbb{R}$. You may assume that these sets form the basis for a topology.

   (a) Prove that 0 is not a limit point of the set $H = \{1/n\}_{n\in\mathbb{N}}$ in the topological space $\mathbb{R}_{\text{har}}$,

   (b) In the topological space $\mathbb{R}_{\text{har}}$, what is the closure of the set $H = \{1/n\}_{n\in\mathbb{N}}$? Prove your answer.

Math 675: Differential Equations

3. Find the solution to

$$\mathbf{x}' = \begin{pmatrix} 1 & 9 \\ 4 & 6 \end{pmatrix} \mathbf{x} + \begin{pmatrix} e^t \\ 13 \end{pmatrix}$$

   Show all work. No technology is allowed except for arithmetic involving fractions.

4. Find two linearly independent solutions to the following differential equation about the origin. Please show at least 4 terms of each solution, if possible.

$$y'' - 2xy' + 6y = 0.$$

Math 680: Optimization

5. Solve the following problem using the Simplex method. Clearly show the set up of the problem, the Simplex tableau, and the solution.

$$\begin{array}{rrrrl} \text{maximize} & 4x_1 & + 8x_2 & + 3x_3 & \\ \text{subject to} & 2x_1 & + 3x_2 & + 2x_3 & \geq 26 \\ & -4x_1 & + 3x_2 & + 3x_3 & \leq 30 \\ & 2x_1 & + 2x_2 & + 5x_3 & \leq 32 \\ & & & x_1, x_2, x_3 & \geq 0 \end{array}$$

6. Suppose that a toy company imports dolls manufactured in Asia. Ships carrying the dolls arrive in either Seattle, San Francisco, or Los Angeles, and then the dolls are transported by truck to distribution centers in Chicago, Houston, New York and Miami. The corresponding transportation problem is given in the following figure, with supplies and demands in circles. The numbers represent thousands of dolls. The costs are printed along the routes, representing hundreds of dollars. Using the techniques of transportation problems, find the cheapest way to ship the dolls from the three shipyards to the four distribution centers.