Department of Mathematics
Comprehensive Examination
2022 Fall Semester "Choose 2" Classes

Directions: You will answer THREE questions from a total of four questions, posed from two classes.

Time: 2 hours

Math 640: Complex Variables

1. Use the Residue Theorem to evaluate

$$\int_0^\infty \frac{x^2}{(x^2+1)^2}\mathrm{d}x$$

   Give full details to establish your result.

2. (a) State Liouville's Theorem

   (b) Let f be an entire function such that $Ref(z) < 1$ for each $z \in \mathbb{C}$. Prove the f is a constant function. (Suggestion: consider $g(z) = e^{f(z)}$)

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Math 660: Topology

3. Recall the following

   Definition: Let $(X, \mathcal{T})$ be a topological space. We say X is **Hausdorff**, or a T_2-**space**, if and only if for every pair x, y of distinct points there are disjoint open sets U, V such that $x \in U$ and $y \in V$.

   Let $(X, \mathcal{T}_{\text{cofinite}})$ denote a set with the cofinite topology. Let $F_n = \{1, 2, 3, ..., n\}$ with $n > 0$ denoting an integer.

   (a) Is $(F_n, \mathcal{T}_{\text{cofinite}})$ a Hausdorff space? Prove your answer.

   (b) Is $(\mathbb{Z}, \mathcal{T}_{\text{cofinite}})$ a Hausdorff space? Prove your answer.

4. Recall the following

   Definition: The square $[0, 1] \times [0, 1]$ with the lexicographic order and its associated order topology is called the **lexicographically ordered square**.

   Let $A = \{(x, 0) \mid 0 < x < 1\}$. Find the closure of A in the lexicographically ordered square. Prove your answer.

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Math 675: Differential Equations

5. Find the solution of the given initial value problem, using Laplace Transforms (refer to the table below).

$$y'' + 2y' + 5y = 0,\ y(0) = 1, y'(0) = 3$$

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$	$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1	$\frac{1}{s},\ s > 0$	e^{at}	$\frac{1}{s-a},\ s > a$
$t^n, n =$ positive integer	$\frac{n!}{s^{n+1}}, s > 0$	$t^p, p > -1$	$\frac{\Gamma(p+1)}{s^{p+1}},\ s > 0$
$\sin(at)$	$\frac{a}{s^2+a^2},\ s > 0$	$\cos(at)$	$\frac{s}{s^2+a^2}.\ s > 0$
$\sinh(at)$	$\frac{a}{s^2-a^2},\ s > \|a\|$	$\cosh(at)$	$\frac{s}{s^2-a^2}.\ s > \|a\|$
$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2},\ s > a$	$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2},\ s > a$
$t^n e^{at}, n =$ positive integer	$\frac{n!}{(s-a)^{n+1}},\ s > a$	$u_c(t)$	$\frac{e^{-cs}}{s},\ s > 0$
$u_c(t)f(t-c)$	$F(s-c)$	$f(ct)$	$\frac{1}{c}F\left(\frac{s}{c}\right).\ c > 0$
$\delta(t-c)$	e^{-cs}	$(-t)^n f(t)$	$F^{(n)}(s)$
$f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$		

6. Solve the following system of ordinary differential equations.

$$\mathbf{x}'(t) = \begin{bmatrix} 1 & -2 & -2 \\ 0 & 3 & 0 \\ 1 & 1 & 4 \end{bmatrix} \mathbf{x}(t)$$

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Math 680: Optimization

7. Solve the following problem using the Simplex method. Clearly show the set up of the problem, the Simplex tableau, and the solution.

$$\begin{array}{lrrrl} \text{maximize} & 4x_1 & +\,3x_2 & -\,5x_3 & \\ \text{subject to} & 2x_1 & -\,x_2 & +\,4x_3 & \leq 16 \\ & 5x_1 & +\,5x_2 & +\,2x_3 & \geq 18 \\ & -3x_1 & +\,4x_2 & +\,2x_3 & = 22 \\ & & & x_1, x_2, x_3 & \geq 0 \end{array}$$

8. Using the Complementary Slackness Theorem, determine whether $(4, 5, 1, 0)$ is the optimal solution to

$$\begin{array}{lrrrrl} \text{minimize} & 5x_1 & -\,6x_2 & +\,11x_3 & +\,8x_4 & \\ \text{subject to} & x_1 & +\,2x_2 & -\,3x_3 & +\,4x_4 & = 11 \\ & 2x_1 & +\,5x_2 & -\,6x_3 & +\,x_4 & \geq 26 \\ & 3x_1 & -\,2x_2 & +\,4x_3 & -\,x_4 & \geq 6 \\ & & & & x_1, x_2, x_3, x_4 & \geq 0 \end{array}$$

Department of Mathematics
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2022 Fall Semester Core Classes

Time: 2.5 hours
Directions:

- You will answer FOUR questions.
- You must choose at least ONE question from each class (one from Math 620, one from Math 630, and one from Math 670).
- The fourth question can come from any of the classes.

Math 620

1. Let G and H be groups and $\varphi : G \to H$ a group homomorphism.
   (a) Prove $\ker \varphi = \{g \in G : \varphi(g) = e_H\}$ is a subgroup of G.
   (b) Assume $B \subseteq H$ is a subgroup of H. Prove $\varphi^{-1}(B) = \{g \in G : \varphi(g) \in B\}$ is a subgroup of G.
   (c) Suppose further that H is a commutative group, $A \subseteq G$ is a subgroup of G , and that $\ker \varphi \subseteq A$. Prove that A is a normal subgroup in G.

2. Let $\varphi : R \to S$ be a surjective ring homomorphism where R and S are rings with unity (by which we mean a multiplicative identity).
   (a) Define the center of a ring A to be the set
   $$Z(A) = \{a \in A : ax = xa \text{ for all } x \in A\}$$
   Prove the image of the center of R is contained in the center of S.
   (b) Prove that if P is a prime ideal in R such that $\ker \varphi \subset P$, then $\varphi(P)$ is a prime ideal in S.

Math 630

3. Let M be a metric space and $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions from M to $\mathbb{R}$ which converges uniformly to a function $f : M \to \mathbb{R}$. Prove that if each f_n is continuous at a point $a \in M$, then f is continuous at a.

4. (a) Suppose E_n for $n \in \mathbb{N}$ are subsets of $\mathbb{R}$ and are each of measure 0. Prove that
   $$\bigcup_{n=1}^{\infty} E_n$$
   is also of measure 0.
   (b) Suppose $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ are both integrable functions on $[a, b]$. Justify why $f + g$ is also integrable on $[a, b]$.

$\Leftarrow$ Turn Over $\Rightarrow$

5. Consider the equation $e^{2x} = 3x + 2$.

   (a) Show that there exist precisely two real solutions of the equation $e^{2x} = 3x + 2$, one negative and one positive.

   (b) Suppose z is the positive solution; find an approximation x of z such that $\|z - x\| < 10^{-6}$

   (c) Show that your approximation x of z is in fact within 10^{-6} of z.

   Note: for this problem, you may not use any graphing or root finding capabilities on your calculator.

6. Let

$$A = \begin{bmatrix} 4 & 0 & -2 \\ 1 & 4 & 0 \\ 0 & 1 & 2 \end{bmatrix}$$

   Suppose we attempt to solve the equation $Ax = b$ by the iterative method

$$\mathbf{x}^{(k)} = M^{-1}N\mathbf{x}^{(k-1)} + M^{-1}\mathbf{b}$$

   where M and N come from the following splitting of A.

$$A = M - N = \begin{bmatrix} \frac{7}{2} & 0 & 0 \\ 0 & \frac{7}{2} & 0 \\ 0 & 0 & \frac{7}{4} \end{bmatrix} - \begin{bmatrix} -\frac{1}{2} & 0 & 2 \\ -1 & -\frac{1}{2} & 0 \\ 0 & -1 & -\frac{1}{4} \end{bmatrix}$$

   Explain why for each $\mathbf{b}$ and for each initial approximation $\mathbf{x}^{(0)}$ the sequence $\{x^{(k)}\}$ converges to the solution of $A\mathbf{x} = \mathbf{b}$.