Department of Mathematics
Comprehensive Examination
2023 Fall Semester "Choose 2" Classes

Directions: You will answer THREE questions from a total of six questions, posed from two classes.

Time: 2 hours

Math 640: Complex Variables

No Complex Variables exam was given in Fall 2023

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Math 660: Topology

4. Recall the following definition

> **Definition:** The **boundary** of A, denoted by ∂A, is defined to be
> $$\overline{A} \cap \overline{X - A}\ .$$

Let $(X, \mathcal{T})$ be a topological space. Is it true that if $\overline{A} \subseteq \overline{B}$ then $\partial A \subseteq \partial B$? Give a full proof or provide a counterexample.

5. Let A be a compact subspace of a Hausdorff space X. Prove that A is closed.

6. Recall the following definitions:

> **Definition:** Let X be a topological space. Then X is **connected** if and only if X is not the union of two disjoint non-empty open sets.

> **Definition:** Let X be a topological space. Subsets A, B in X are **separated** if and only if $\overline{A} \cap B = A \cap \overline{B} = \emptyset$. Thus B does not contain any limit points of A, and A does not contain any limit points of B.

Prove that X is connected if and only if X is **not** the union of two disjoint non-empty separated sets.

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Math 675: Differential Equations

7. Consider the system

$$\mathbf{x}'(t) = \begin{pmatrix} 1 & -4 \\ 1 & 1 \end{pmatrix} \mathbf{x}(t)$$

   (a) Find the general solution to the system. Express your answer in terms of real valued functions.

   (b) Determine whether solutions rotate clockwise or counterclockwise. Explain your process.

8. Consider the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & -3 \\ -3 & 1 \end{pmatrix}$$

   The general solution to the homogeneous system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ is

$$\mathbf{x}_{hom} = c_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-2t} \qquad (*)$$

   (a) What are the eigenvalues and eigenvectors for $\mathbf{A}$? Explain.

   (b) Calculate the linear propagator $\exp(t\mathbf{A})$ using part (a).

   (c) Use Eq. ($\star$) to find the general solution to the inhomogeneous system:

$$\mathbf{x}'(t) = \begin{pmatrix} 1 & -4 \\ 1 & 1 \end{pmatrix} \mathbf{x}(t) + \begin{pmatrix} e^{4t} \\ 2e^{4t} \end{pmatrix}$$

   Simplify your answer as much as possible.

9. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function. Consider the vector valued functions

$$\mathbf{x}(t) = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \quad \mathbf{y}(t) = \begin{pmatrix} 2f(t) \\ 3f(t) \end{pmatrix}$$

   where $f(t) > 0$ and $f'(t) > 0$ for all $t \in (-\infty, \infty)$.

   (a) Fix $t_0 \in (-\infty, \infty)$. Show that the set $\mathcal{S}(t) = \{\mathbf{x}(t), \mathbf{y}(t)\}$ is linearly **dependent** at each point $t = t_0$.

   (b) Prove, directly from the definition, that the set

$$\mathcal{S}(t) = \{\mathbf{x}(t), \mathbf{y}(t)\}$$

   is linearly independent on the interval $(-\infty, \infty)$.

   (c) Calculate the Wronskian $W[\mathbf{x}(t), \mathbf{y}(t)]$.

   (d) Could $\mathbf{x}(t)$ and $\mathbf{y}(t)$ be solutions to the same 2D linear system of ODE $\mathbf{x}' = \mathbf{A}\mathbf{x}$? Explain your thinking.

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Math 680: Optimization

10. Solve the following problem using the Simplex method. Clearly show the set up of the problem, the Simplex tableaux, and state the solution. You can use either fractions or decimals (rounded to the nearest hundredth).

$$\begin{array}{rrrrl} \text{maximize} & 6x_1 & +\,8x_2 & +\,9x_3 & \\ \text{subject to} & 2x_1 & +\,4x_2 & +\,2x_3 & \geq 11 \\ & 3x_1 & -\,3x_2 & +\,4x_3 & \leq 28 \\ & -x_1 & +\,2x_2 & +\,3x_3 & \leq 30 \\ & x_1 & +\,2x_2 & +\,x_3 & = 32 \\ & & & x_1, x_2, x_3 & \geq 0 \end{array}$$

11. Prove or disprove, using the Complementary Slackness Theorem, that (7,0,3) is the optimal solution to

$$\begin{array}{rrrrl} \min & 5x_1 & +\,6x_2 & +\,8x_3 & \\ \text{subject to} & x_1 & +\,4x_2 & +\,3x_3 & \leq 16 \\ & 2x_1 & +\,3x_2 & -\,2x_3 & \geq 8 \\ & 3x_1 & +\,5x_2 & +\,2x_3 & \leq 26 \\ & & & x_j & \geq 0 \end{array}$$

12. Consider the problem

$$\begin{array}{rrrrl} \min & 6x_1 & +\,8x_2 & +\,3x_3 & \\ \text{subject to} & 2x_1 & +\,x_2 & -\,x_3 & \leq 24 \\ & -3x_1 & -\,4x_2 & +\,2x_3 & \geq 22 \\ & x_1 & +\,2x_2 & +\,3x_3 & \leq 30 \\ & 5x_1 & -\,3x_2 & +\,x_3 & \geq 26 \\ & & & x_j & \geq 0 \end{array}$$

The first and final tableaux are given in the following table.

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	
x_4	2	1	-1	1	0	0	0	24
a_1	-3	4	2	0	-1	0	0	22
x_6	1	2	3	0	0	1	0	30
x_7	5	-3	1	0	0	0	1	26
	6	8	3	0	0	0	0	0
x_4	$\frac{37}{8}$	0	0	1	$\frac{5}{8}$	$\frac{3}{4}$	0	$\frac{131}{4}$
x_2	$-\frac{11}{8}$	1	0	0	$\frac{3}{8}$	$-\frac{1}{4}$	0	$\frac{3}{4}$
x_3	$\frac{5}{4}$	0	1	0	$\frac{1}{4}$	$\frac{1}{2}$	0	$\frac{19}{2}$
x_7	$-\frac{3}{8}$	0	0	0	$-\frac{11}{8}$	$-\frac{5}{4}$	1	$\frac{75}{4}$
	$\frac{53}{4}$	0	0	0	$\frac{9}{4}$	$\frac{1}{2}$	0	$\frac{69}{2}$

Answer the following questions using sensitivity analysis. Treat each situation separately, as it refers to the original problem.

(a) What is the range on the coefficient, c_2, of x_2 in the objective function such that the solution values for (x_1, x_2, x_3) will not change.

(b) What is the result when the right hand side constant on the second constraint changes from 22 to 16. Thus, the second constraint becomes

$$-3x_2 - 4x_2 + 2x_3 \geq 16$$

Department of Mathematics
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2023 Fall Semester Core Classes

Time: 2.5 hours
Directions:

- You will answer FOUR questions.
- You must choose at least ONE question from each class (one from Math 620, one from Math 630, and one from Math 670).
- The fourth question can come from any of the classes.

Math 620

1. Consider two groups G, K and $\varphi : G \to K$ a group homomorphism.
   (a) Prove that the set $\ker \varphi = \{g \in G : \varphi(g) = e_K\}$ is a subgroup of G.
   (b) Assume N is a normal subgroup of K and consider the set
   $$M = \varphi^{-1}(N) = \{g \in G : \varphi(g) \in N\}$$
   Assuming M is a subgroup of G, prove that M is normal and that $\ker \varphi \subset M$.

2. Suppose G is a group, and H a normal subgroup of G.
   (a) Let $g \in G$ and consider a coset $gH = \{gh : h \in H\}$ in the quotient group G/H. Prove $(gH)^k = g^k H$ for all $k \in \mathbb{N}$.
   (b) Suppose G is a cyclic group. Prove that a quotient group $G/H = \{gH : g \in G\}$ is a cyclic group.

3. Let R and S be rings with identity. Recall, the center of the ring R is the set
   $$Z(R) = \{r \in R : rx = xr \text{ for all } x \in R\}$$
   (a) Prove $Z(R)$ is a subring of R.
   (b) Let $a \in R$ and define the centralizer of a in R as the set
   $$C(a) = \{r \in R : ar = ra\}$$
   Prove $Z(R) = \bigcap_{a \in R} C(a)$
   (c) Let $\varphi : R \to S$ be a surjective ring homomorphism. Prove that the image of the center of R under φ, $\mathrm{im}(\varphi(Z(R)) = \{\varphi(r) : r \in Z(R)\}$, is contained, as a subset, in $Z(S)$.

Math 630

4. Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Prove that if f is Riemann integrable, then for every $\epsilon > 0$, there exists a subdivision P_ϵ of $[a, b]$ such that

$$U(f, P_\epsilon) - L(f, P_\epsilon) < \epsilon.$$

5. Let $A \subseteq M$ where M is a metric space. Prove that if the complement of A is open, then for every convergent sequence $\{a_n\}$ of points in A, we have $\lim_{n\to\infty} a_n \in A$.

6. Let $A \subseteq M$ where M is a metric space.
   (a) State the definition of what it means for the set A to be connected.
   (b) Use the definition you have stated to determine and justify whether the sets below are connected or not.
      i. $[-1, 1]$
      ii. $(0, 1) \cup \{2\}$

Math 670

7. Consider the equation $f(x) = e^x - 2x^2$.
   (a) Prove that the equation has exactly three real solutions.
   (b) Let α be the largest of the three solutions. Use Newton's Method to find an approximation of α with an absolute error of less than 10^{-5}.

8. (a) Starting from Taylor series prove that

$$f''(x) = \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}$$

   is a second-order approximation of the second derivative of $f(x)$.
   (b) Clearly describe what "second-order approximation" means in this context.

9. (a) Explain how to solve the equation $A\mathbf{x} = \mathbf{b}$ using an LU decomposition.
   (b) Write the matrix form you would use to numerically solve the following partial differential equation using backward-difference in time and central-difference in space.

$$\frac{\partial u}{\partial t} - \frac{1}{16}\frac{\partial^2 u}{\partial x^2} = 0, \qquad 0 < x < 1, 0 < t;$$

$$u(0,t) = u(1,t) = 0, \qquad u(x,0) = 2\sin(2\pi x)$$